CONE CUBIC CONFIGURATIONS OF A RULED SURFACE*

BY

A. F. CARPENTER

I. Introduction

In a series of papers presented to the San Francisco Section of the American Mathematical Society† during the year 1923, the author has discussed numerous characteristics of certain point and line figurations connected with each line element of the general ruled surface. The definitions and theorems of these papers have been built up by the method of the projective differential geometry of Wilczynski from a few fundamental and well known projective properties of ruled surfaces, chief among which are such notions as the flecnode curve and flecnode surface, the complex curve, the osculating quadric and osculating linear complex.

Before proceeding with the present discussion it will be advisable to restate briefly certain of the definitions and theorems involved.

The flecnode curve C_F of a ruled surface S cuts each line element g of S in two points. The planes osculating C_F at two such points intersect in a line h. For each such line there is determined a second line h', the polar reciprocal of h with respect to the linear complex L osculating S along g. The three lines g, h, h' are in general non-intersecting and hence determine a quadric Q_1. The complete intersection of Q_1 and the osculating quadric Q is made up of the element g of S and a space cubic. This curve C_{F} we call the *primary flecnode cubic.*

To each osculating plane of C_F there corresponds by means of L a point in that plane. The locus of these points for all the osculating planes of C_F is a second space cubic. This curve $C_{F'}$ we call the *secondary flecnode cubic.*

By making use of the complex curve rather than the flecnode curve we arrive at two other cubics called respectively the *primary* and *secondary complex cubics.*

Each space cubic determines a linear complex. The complex L_1 determined by C_F we call the *first associated linear complex.* It develops that C_F *and* $C_{F'}$ *determine the same linear complex.* The linear congruence Γ_1 common to L and L_1 is called the *first associated linear congruence.*

* Presented to the Society, San Francisco Section, April 5, 1924.

† April, September, December, 1923. First paper under the title *Flecnodal properties of a ruled surface* printed in the Tôhoku Mathematical Journal, vol. 23, Nos. 1–2, Aug., 1923. Second paper to appear in the American Journal of Mathematics.

In a similar manner we obtain from the primary and secondary complex cubics a *second associated linear complex* and a *second associated linear congruence*.

As system of reference in the projective differential geometry of ruled surfaces there is frequently used the tetrahedron $P_y\,P_z\,P_\varrho\,P_\sigma$. Of the four vertices of this tetrahedron P_y and P_z are the flecnode points of $l_{yz} \equiv g$, and P_ϱ, P_σ are two points, one on each of the flecnode tangents $l_{y\varrho}$, $l_{z\sigma}$, drawn to S at P_y, P_z respectively. Unit point in this system is so chosen that the coördinates of the four vertices are P_y, (1, 0, 0, 0); P_z, (0, 1, 0, 0); P_ϱ, (0, 0, 1, 0); P_σ, (0, 0, 0, 1). Of the six edges of the tetrahedron four, namely l_{yz}, $l_{\varrho\sigma}$, $l_{y\sigma}$, $l_{z\varrho}$, belong to L. It follows that the planes which correspond to the points of any one of these lines by means of L constitute a pencil of planes on that line as an axis. There are thus determined four axial pencils.

If, between the points of l_{yz} and $l_{z\varrho}$, there is set up a one-to-one projective correspondence in which to the general point $(\alpha, \beta, 0, 0)$ of l_{yz} there corresponds the point $(0, \alpha, \beta, 0)$ of $l_{z\varrho}$, then there is likewise set up a one-to-one projective correspondence between the planes of the two pencils on these two lines. Since the axes of these two projective pencils intersect, the lines of intersection of corresponding pairs of planes must have for their locus a quadric cone K_1 whose vertex is at P_z. By making use of the pairs of lines $(l_{z\varrho}, l_{\varrho\sigma})$, $(l_{yz}, l_{y\sigma})$, $(l_{y\sigma}, l_{\varrho\sigma})$, in a similar way we define three other cones K_2, K_3, K_4. These four quadric cones we call the *complex cones associated with g*.

Since the pair of lines $(l_{yz}, l_{\varrho\sigma})$ do not intersect, the two pencils of planes on these lines determine a non-developable quadric. It proves to be in fact the osculating quadric, Q. The other pair of non-intersecting lines $(l_{z\varrho}, l_{y\sigma})$ also determines a quadric, Q'; Q and Q' we call the *complex quadrics*. The equations of the four complex cones and the two complex quadrics, in this system of coördinates, are

$$
(1)\quad
\begin{aligned}
&(K_1)\ p_{12}^2x_1x_3+p_{21}^2x_4^2=0, \quad (K_2)\ p_{12}^2x_1^2+p_{21}^2x_2x_4=0, \quad (K_3)\ p_{12}^2x_3^2+p_{21}^2x_2x_4=0,\\
&(K_4)\ p_{12}^2x_1x_3+p_{21}^2x_2^2=0, \quad (Q)\quad x_1x_4-\ \ x_2x_3=0, \quad (Q')\quad x_1x_2-\ \ x_3x_4=0,
\end{aligned}
$$

where p_{12}, p_{21} are two of the coefficients of the system of differential equations defining the ruled surface S.

The four complex cones K_1, K_2, K_3, K_4 can be paired in six ways. For each of the four pairs $(K_1\,K_2)$, $(K_3\,K_4)$, $(K_1\,K_3)$, $(K_2\,K_4)$, the complete intersection is composed of a straight line and a space cubic. These four curves C_1, C_2, C_3, C_4 we call the *primary cone cubics associated with g*. For each of these cubics we indicate below its equations in parametric

form, the two cones upon which it lies and the line which completes their intersection:

$$
(2)\quad
\begin{array}{llllll}
(C_1) & x_1 = -p_{21}^2 t, & x_2 = -p_{21}^2, & x_3 = p_{12}^2 t^3, & x_4 = p_{12}^2 t^2; & (K_1 K_2),\ l_{z\varrho}; \\
(C_2) & x_1 = p_{12}^2 t^3, & x_2 = p_{12}^2 t^2, & x_3 = -p_{21}^2 t, & x_4 = -p_{21}^2; & (K_3 K_4),\ l_{y\sigma}; \\
(C_3) & x_1 = p_{12}^2 t^3, & x_2 = -p_{21}^2, & x_3 = -p_{21}^2 t, & x_4 = p_{12}^2 t^2; & (K_1 K_3),\ l_{yz}; \\
(C_4) & x_1 = -p_{21}^2 t, & x_2 = p_{12}^2 t^2, & x_3 = p_{12}^2 t^3, & x_4 = -p_{21}^2; & (K_2 K_4),\ l_{\varrho\sigma}.
\end{array}
$$

By making use of their osculating planes and the points corresponding to them by means of L, we obtain, from the four cubics $C_1, \cdots, C_4$, four new cubics $C_1', \cdots, C_4'$, just as the secondary flecnode cubic is obtained from the primary flecnode cubic. These four curves $C_1', \cdots, C_4'$ we call the *secondary cone cubics associated with g*. Their equations in parametric form are

$$
(3)\quad
\begin{array}{lllll}
(C_1') & x_1 = p_{21}^3, & x_2 = 3p_{12}p_{21}^2 t, & x_3 = 3p_{12}^2 p_{21} t^2, & x_4 = p_{12}^3 t^3; \\
(C_2') & x_1 = 3p_{12}^2 p_{21} t^2, & x_2 = p_{12}^3 t^3, & x_3 = p_{21}^3, & x_4 = 3p_{12}p_{21}^2 t; \\
(C_3') & x_1 = 3p_{12}^2 p_{21} t^2, & x_2 = -3p_{12}p_{21}^2 t, & x_3 = p_{21}^3, & x_4 = -p_{12}^3 t^3; \\
(C_4') & x_1 = p_{21}^3, & x_2 = -p_{12}^3 t^3, & x_3 = 3p_{12}^2 p_{21} t^2, & x_4 = -3p_{12}p_{21}^2 t.
\end{array}
$$

The parametric equations of the primary and secondary flecnode cubics, and the equation, in line coördinates, of the linear complex L_1 which they determine, are respectively

$$
(4)\quad
\begin{array}{lll}
(C_F) & x_1 = 2t(p_{12}^2 q_{21} t^2 - p_{21}^2 q_{12}), & x_2 = 2(p_{12}^2 q_{21} t^2 - p_{21}^2 q_{12}), \\
& x_3 = p_{12} p_{21} t (p_{12} t^2 - p_{21}), & x_4 = p_{12} p_{21} (p_{12} t^2 - p_{21}); \\
(C_{F'}) & x_1 = 2p_{21}(3p_{12}^2 q_{21} t^2 + p_{21}^2 q_{12}), & x_2 = 2p_{12} t (p_{12}^2 q_{21} t^2 + 3p_{21}^2 q_{12}), \\
& x_3 = p_{12} p_{21}^2 (3p_{12} t^2 + p_{21}), & x_4 = p_{12}^2 p_{21} t (p_{12} t^2 + 3p_{21});
\end{array}
$$

$$
(L_1)\quad 2p_{12}p_{21}\omega_{12} - (p_{12}q_{21} + 3p_{21}q_{12})\omega_{14} + (3p_{12}q_{21} + p_{21}q_{12})\omega_{23} + 8q_{12}q_{21}\omega_{34} = 0;
$$

where q_{12}, q_{21}, are another pair of coefficients of the system of differential equations defining S.

The equation of the quadric Q_1 is

$$
(Q_1)\quad p_{12}^2 p_{21} x_1 x_3 - p_{12} p_{21}^2 x_2 x_4 - 2p_{12}^2 q_{21} x_3^2 + 2p_{21}^2 q_{12} x_4^2 = 0.
$$

If we identify the parameters in the equations of the ten cubics listed above we thereby set up a point correspondence between these curves. But this correspondence is not arbitrary. That between C_F and $C_{F''}$ is

indeed the one resulting from the definition of $C_{F''}$. The projective transformation*

$$(5_1)\qquad \begin{aligned} \bar{x}_1 &= -p_{21}x_1 + 2q_{21}x_3, & \bar{x}_2 &= -p_{21}x_2 + 2q_{21}x_4,\\ \bar{x}_3 &= p_{12}x_1 - 2q_{12}x_3, & \bar{x}_4 &= p_{12}x_2 - 2q_{12}x_4 \end{aligned}$$

carries $C_{F'}$ into C_1 and $C_{F''}$ into C_1'. An examination of equations (2) shows that the four curves $C_1, \cdots, C_4$ are projectively equivalent, the projectivities which carry C_1 into C_2, C_3, C_4 being respectively

$$(5_2)\qquad \bar{x}_1 = x_3,\quad \bar{x}_2 = x_4,\quad \bar{x}_3 = x_1,\quad \bar{x}_4 = x_2;$$

$$(5_3)\qquad \bar{x}_1 = x_3,\quad \bar{x}_2 = x_2,\quad \bar{x}_3 = x_1,\quad \bar{x}_4 = x_4;$$

$$(5_4)\qquad \bar{x}_1 = x_1,\quad \bar{x}_2 = x_4,\quad \bar{x}_3 = x_3,\quad \bar{x}_4 = x_2.$$

Similarly for the curves $C_1', \cdots, C_4'$, but with a different set of projectivities. The point correspondence between the five curves $C_{F'}$, C_1, C_2, C_3, C_4, as well as that between the five curves $C_{F''}$, C_1', C_2', C_3', C_4', is therefore projective in nature, while that between the pairs C_j, C_j' $(j = 1, \cdots, 4)$, is exactly that determined by the complex L between $C_{F'}$ and $C_{F''}$.

From the general theory† a fundamental system of simultaneous solutions $y_k(x)$, $z_k(x)$ $(k = 1, \cdots, 4)$ of the system of equations defining S determines two directrix curves C_y, C_z of S. Corresponding points P_y, P_z of these curves have the homogeneous coördinates (y_1, y_2, y_3, y_4), (z_1, z_2, z_3, z_4) and lie upon the same generator g of S. For our choice of a coördinate system, as we have seen, $y_1 = 1$, $y_2 = y_3 = y_4 = 0$; $z_2 = 1$, $z_1 = z_3 = z_4 = 0$; while $\varrho_3 = 1$, $\varrho_1 = \varrho_2 = \varrho_4 = 0$; $\sigma_4 = 1$, $\sigma_1 = \sigma_2 = \sigma_3 = 0$; where

$$(6)\qquad \varrho = 2\frac{dy}{dx} + p_{12}z, \qquad \sigma = 2\frac{dz}{dx} + p_{21}y.$$

For the general point P_x; (a, b, c, d), the coördinates are given by the expression

$$x_k = a y_k + b z_k + c\varrho_k + d\sigma_k \qquad (k = 1, \cdots, 4),$$

the subscript k ordinarily being omitted.

---

* p_{12}, p_{21}, q_{12}, q_{21} are functions of a parameter x which is constant so long as we are considering a single line element g of S. Hence this transformation is projective for all configurations associated with g.

† *Projective Differential Geometry of Curves and Ruled Surfaces*, Wilczynski; B. G. Teubner, 1906, pp. 129, 130. Hereafter referred to as Proj. Dif. Geom.

It is the purpose of this paper to characterize with reasonable completeness the configurations formed by the ten curves $C_{F'}, C_{F''}; C_1, \cdots, C_4; C_1', \cdots, C_4'$, to define and discuss two additional sets of cubics, and to suggest further problems for investigation.

II. The first and second cone rays

There are four pencils of quadrics determined by the pairs of cones $(K_1 K_2)$, $(K_3 K_4)$, $(K_1 K_3)$, $(K_2 K_4)$, namely

$$(7) \qquad \begin{aligned} &h(p_{12}^2 x_1 x_3 + p_{21}^2 x_4^2) + k(p_{12}^2 x_1^2 + p_{21}^2 x_2 x_4) = 0,\\ &h(p_{12}^2 x_3^2 + p_{21}^2 x_2 x_4) + k(p_{12}^2 x_1 x_3 + p_{21}^2 x_2^2) = 0,\\ &h(p_{12}^2 x_1 x_3 + p_{21}^2 x_4^2) + k(p_{12}^2 x_3^2 + p_{21}^2 x_2 x_4) = 0,\\ &h(p_{12}^2 x_1^2 + p_{21}^2 x_2 x_4) + k(p_{12}^2 x_1 x_3 + p_{21}^2 x_2^2) = 0. \end{aligned}$$

By using the same parameters in each of the four equations (7) we have set up an arbitrary correspondence between the quadrics of these four pencils. We do this as a matter of convenience.

C_1 lies upon each of the quadrics of (7_1). Since C_1 also lies upon Q each of these quadrics (7_1) must have in common with Q one, and only one, straight line. It proves to be the line joining the point $hy-k\varrho$ on $l_{y\varrho}$ to the point $hz-k\sigma$ on $l_{z\sigma}$. As we pass from surface to surface of the pencil (7_1), this line coincides successively with all the rulings of Q.

C_2 lies upon each of the quadrics of (7_2). Since C_2 also lies upon Q, each of these quadrics (7_2) must have in common with Q one, and only one, straight line. It also proves to be the line determined by the points $hy-k\varrho$ and $hz-k\sigma$, so that corresponding quadrics of the pencils (7_1) and (7_2) cut Q in the same line.

The cubics C_3, C_4 are seen by inspection to lie upon the quadric Q'. But C_3 lies upon each quadric of the pencil (7_3) and C_4 upon each quadric of the pencil (7_4). Each quadric of (7_3), as also each of (7_4), must therefore have one, and only one, straight line in common with Q'. This line is seen to be that which is determined by the two points $ky-h\varrho$, $hz-k\sigma$ on $l_{y\varrho}$, $l_{z\sigma}$, respectively. Corresponding pairs of quadrics of the two pencils (7_3), (7_4) have this line in common with Q'. As we pass from surface to surface of (7_3) or (7_4), this line coincides successively with all of the rulings of Q'.

If we write for the general point on the first of these lines the expression

$$f(hy-k\varrho)+g(hz-k\sigma),$$

and, for the general point on the second, the expression

$$m(ky-h\varrho)+n(hz-k\sigma),$$

then it follows that these two points will coincide if and only if

$$\frac{h}{k}=\frac{k}{h}, \text{ that is, if } h=\pm k.$$

One may thus in two ways choose a set of four quadrics, one from each pencil, such that the four will have a line in common and this line will at the same time lie upon both the complex quadrics. The points in which these two lines cut the flecnode tangents $l_{y\varrho}$, $l_{z\sigma}$ will engage our attention again. They are given by the expressions

$$(8) \qquad \alpha=y-\varrho, \quad \beta=z-\sigma; \qquad \gamma=y+\varrho, \quad \delta=z+\sigma.$$

We note that P_α, P_γ *are harmonic conjugates with respect to* P_y, P_ϱ, *as are also* P_β, P_δ, *with respect to* P_z, P_σ.

The cones K_1 and K_2 have C_1 and $l_{z\varrho}$ as their complete intersection. The line $l_{z\varrho}$ is tangent to K_3 at P_z but is not an element of K_3. If we substitute from (C_1) of (2) into (K_3) of (1) we find for t the six values

$$0, \quad 0, \quad p_{21}/p_{12}, \quad -p_{21}/p_{12}, \quad p_{21}i/p_{12}, \quad -p_{21}i/p_{12}.$$

The first two of these correspond to the point P_z and the remaining four, by means of (C_1), give respectively the four points P_φ, P_ψ, P_η, P_ζ, where

$$(9) \qquad \begin{aligned} \varphi &= p_{21}\,y+p_{12}z-p_{21}\,\varrho-p_{12}\sigma = p_{21}\,\alpha+p_{12}\beta,\\ \psi &= p_{21}\,y-p_{12}z-p_{21}\,\varrho+p_{12}\sigma = p_{21}\,\alpha-p_{12}\beta,\\ \eta &= p_{21}iy+p_{12}z+p_{21}i\varrho+p_{12}\sigma = p_{21}i\gamma+p_{12}\delta,\\ \zeta &= p_{21}iy-p_{12}z+p_{21}i\varrho-p_{12}\sigma = p_{21}i\gamma-p_{12}\delta. \end{aligned}$$

From (K_4) of (1), the point P_z is not on K_4, but the four points P_φ, P_ψ, P_η, P_ζ are. Moreover, (9) shows that these four points are also on Q and Q'. Two of them, P_φ, P_ψ, are in fact harmonic conjugates with respect to P_α, P_β, and the other two, P_η, P_ζ, harmonic conjugates with respect to P_γ, P_δ. These four points P_φ, P_ψ, P_η, P_ζ, are the only ones common to all four cones $K_1, \cdots, K_4$. Since they may be thought of as determining the two lines $l_{\alpha\beta}$, $l_{\gamma\delta}$, we shall speak of these two lines as the *first and second cone rays associated with* g.

The four points P_φ, P_ψ, P_η, P_ζ have already been shown to lie on C_1. By introducing the values $t = \pm p_{21}/p_{12}$, $\pm p_{21} i/p_{12}$ into (C_2), (C_3), (C_4) of (2) *it results that all four curves pass through these four points and that in the point correspondence existing between the four curves, these four points are self-corresponding.* It should be noted that P_φ, thought of as being on C_1, C_2, corresponds to P_ψ, thought of as being on C_3, C_4, and conversely, while P_η counts for a complete set of four corresponding points, as does also P_ζ.

All four points cannot be real at the same time, the two on the second cone ray, $l_{\gamma\delta}$, being imaginary if those on the first cone ray, $l_{\alpha\beta}$, are real, and conversely. It may happen that none of the four are real. We shall speak of these four points as the *focal points of the primary cone cubics.*

If (f, g, h, k) be any point, then the three points whose coördinates are (h, k, f, g), (h, g, f, k), (f, k, h, g), as well as the three points whose coördinates are (h, k, f, g), $(h, -g, f, -k)$, $(f, -k, h, -g)$, determine a plane on which the first point lies. It follows from equations (5) that if P_1, P_2, P_3, P_4 are a set of four corresponding points, one on each of the curves C_1, C_2, C_3, C_4, then P_1, P_2, P_3, P_4 are coplanar. Likewise it follows that if P_1', P_2', P_3', P_4' are a set of four corresponding points, one on each of the curves C_1', C_2', C_3', C_4', then P_1', P_2', P_3', P_4' are coplanar. The equations of the planes p and p' containing these two sets of points are easily found. For the first set we find the equation

$$x_1 + x_3 - t(x_2 + x_4) = 0, \tag{10}$$

and for the second set, the equation

$$p_{12}\varkappa t(x_1 - x_3) - p_{21}\lambda(x_2 - x_4) = 0, \tag{11}$$

where

$$\varkappa = p_{12}^2 t^2 - 3p_{21}^2, \qquad \lambda = 3p_{12}^2 t^2 - p_{21}^2. \tag{12}$$

The points $\alpha = y - \varrho$, $\beta = z - \sigma$ are on the first of these two planes and the points $\gamma = y + \varrho$, $\delta = z + \sigma$ are on the second. It follows that equations (10) and (11) define two pencils of planes whose axes are the first and second cone rays, t being the parameter of the pencil in each case. In brief, *the projectivity existing between the points of the four primary (secondary) cone cubics is such that corresponding points lie by fours in the planes of an axial pencil whose axis is the first (second) cone ray.* We note from (4), (10) and (11) that the point on the primary flecnode cubic which corresponds to the set of points $P_1, \cdots, P_4$ lies on their plane, but that the point of the secondary flecnode cubic which corresponds to the set $P_1', \cdots, P_4'$ lies on their plane only if $p_{12} - 2q_{12} = p_{21} - 2q_{21}$.

III. Perspectivities of the cone cubics

Since the cubics C_1 and C_2 lie on Q while the cubics C_3 and C_4 lie on Q', it results that of the four points P_1, P_2, P_3, P_4, in which these four curves cut the plane p, P_1 and P_2 lie on Q and P_3 and P_4 upon Q'. Since $l_{\alpha\beta}$ lies on both Q and Q', the plane p through this line must be a tangent plane to both quadrics. Its points of tangency with Q and Q' are given by the respective expressions

$$\tau = t\alpha + \beta, \qquad \tau' = -t\alpha + \beta. \tag{13}$$

These points are harmonic conjugates with respect to P_α and P_β. We note that the points P_1, P_2 are not on $l_{\alpha\beta}$. They must therefore lie upon the second line which p has in common with Q, so that *P_1, P_2 and P_τ are collinear*. A similar argument leads to the conclusion that *P_3, P_4 and $P_{\tau'}$ are collinear*. We find further that *P_1, P_3 and P_α are collinear as are also the sets P_2, P_4, P_α; P_1, P_4, P_β; P_2, P_3, P_β*.

The lines P_1P_2 and P_3P_4, lying in plane p, intersect. But P_1P_2 is a ruling of Q and P_3P_4 a ruling of Q'. Their point of intersection must therefore be a point of the intersection of Q and Q'. The complete intersection of these two quadrics consists of the two flecnode tangents $l_{y\varrho}$, $l_{z\sigma}$, and the two cone rays $l_{\alpha\beta}$, $l_{\gamma\delta}$. Moreover the flecnode tangents belong to one regulus and the cone rays to the other regulus, on each quadric. Since P_1P_2 and P_3P_4 intersect $l_{\alpha\beta}$ in distinct points, their point of intersection must be on $l_{\gamma\delta}$. It is indeed the point given by the expression

$$\theta = t\gamma + \delta.$$

To sum up, we find that *the complete quadrangle whose vertices $P_1, \cdots, P_4$ are any set of corresponding primary cone cubic points lies on a plane which is at the same time tangent to both of the complex quadrics Q and Q'. Of its six sides, one, P_1P_2, lies on Q and another, P_3P_4, lies on Q'. Of its diagonal points, two, P_α, P_β, lie upon the first cone ray and are the same for all such quadrangles, while the third lies upon the second cone ray.*

That plane of the pencil (10) which is tangent to Q at the point P_{τ_1}, where $\tau_1 = t_1\alpha + \beta$, is given by the equation

$$x_1 + x_3 - t_1(x_2 + x_4) = 0. \tag{10_1}$$

The tangent plane to Q' at this same point has for its equation

$$x_1 + x_3 + t_1(x_2 + x_4) = 0. \tag{10_{-1}}$$

But the plane (10_1) is tangent to Q' at $P_{\tau_{-1}}$, where $\tau_{-1} = -t_1\alpha+\beta$, and the plane (10_{-1}) is tangent to Q at this same point.

Let us think of the planes of the pencil (10) as paired in this way, the two planes of such a pair being indicated by p_1 and p_{-1} and the quadrangles in these planes by $(P_1P_2P_3P_4)$ and $(P_{-1}P_{-2}P_{-3}P_{-4})$. Then it is seen that in either of the two orders

$$(P_1P_2P_3P_4)\sim(P_{-3}P_{-4}P_{-1}P_{-2}) \text{ or } (P_1P_2P_3P_4)\sim(P_{-4}P_{-3}P_{-2}P_{-1}),$$

we can determine a correspondence between these two quadrangles such that corresponding pairs of lines intersect in $l_{\alpha\beta}$. We find indeed that the lines P_1P_3, P_2P_4, $P_{-1}P_{-3}$, $P_{-2}P_{-4}$ intersect in P_α, the lines P_1P_4, P_2P_3, $P_{-1}P_{-4}$, $P_{-2}P_{-3}$ intersect in P_β, the lines P_1P_2, $P_{-3}P_{-4}$, in the point P_τ and the lines P_3P_4, $P_{-1}P_{-2}$, in the point $P_{\tau'}$.

It follows that in two ways these two quadrangles are in perspective from a point. It can be easily verified that the four lines P_1P_{-3}, P_2P_{-4}, P_3P_{-1}, P_4P_{-2} pass through the point P_γ, while the lines P_1P_{-4}, P_2P_{-3}, P_3P_{-2}, P_4P_{-1} pass through the point P_δ.

Without further discussion we remark that if the planes of the pencil on the second cone ray $l_{\gamma\delta}$ are paired as above, so that of the two in each pair, the first, q_1,

$$x_1-x_3-t_1(x_2-x_4) = 0,$$

is tangent to Q at $\theta_1 = t_1\gamma+\delta$, while the second, q_{-1},

$$x_1-x_3+t_1(x_2-x_4) = 0,$$

is tangent to Q' at the same point, then the quadrangle $(P_1P_2P_{-3}P_{-4})$ lies in the first of these planes, the quadrangle $(P_3P_4P_{-1}P_{-2})$ lies in the second, and these two quadrangles are in perspective from the two points P_α, P_β, the four lines P_1P_3, P_2P_4, $P_{-3}P_{-1}$, $P_{-4}P_{-2}$ passing through P_α and the four lines P_1P_4, P_2P_3, $P_{-3}P_{-2}$, $P_{-4}P_{-1}$ through P_β.

To recapitulate: let those points of the first and second cone rays $l_{\alpha\beta}$, $l_{\gamma\delta}$ correspond which lie upon the same line of that regulus of Q to which the flecnode tangents belong. At each of such a pair of corresponding points P_τ, P_θ, construct the tangent planes to Q and Q'. Of these four planes, two, p_1, p_{-1}, are on $l_{\alpha\beta}$, and two, q_1, q_{-1}, are on $l_{\gamma\delta}$. Of the total of $3\cdot4\cdot4 = 48$ points of intersection of these four planes with the four primary cone cubics, 32 coincide by eights in the four focal points P_φ, P_ψ, P_η, P_ζ. The remaining 16 points coincide by twos.

Of these eight distinct points four lie in each of the planes p_1, p_{-1}, q_1, q_{-1}, in such a way that no plane contains more than one point from each cone cubic. Consider the points P_α, P_β, and P_γ, P_δ, in which $l_{\alpha\beta}$ and $l_{\gamma\delta}$ are cut by $l_{\alpha\gamma}$ and $l_{\beta\delta}$. The lines joining either P_γ or P_δ to the four cone cubic points in either p_1 or p_{-1} pass through the cone cubic points in the other of these two planes, and the lines joining either P_α or P_β to the four cone cubic points in either q_1 or q_{-1} pass through the four cone cubic points in the other of these two planes.

We have already seen that, for each set of corresponding points P_1, P_2, P_3, P_4, lying in the plane p_1, there exists a set of corresponding secondary cone cubic points P'_1, P'_2, P'_3, P'_4, lying in the plane p'_1,

$$(11_1) \qquad p_{12}\varkappa_1 t_1(x_1-x_3)-p_{21}\lambda_1(x_2-x_4) = 0.$$

Similarly, for the set of points P_{-1}, P_{-2}, P_{-3}, P_{-4}, lying in the plane p_{-1}, there is a set P'_{-1}, P'_{-2}, P'_{-3}, P'_{-4}, lying in the plane p'_{-1},

$$(11_{-1}) \qquad p_{12}\varkappa_1 t_1(x_1-x_3)+p_{21}\lambda_1(x_2-x_4) = 0.$$

For these two quadrangles lying in the planes p'_1, p'_{-1}, it also holds that in either of two orders,

$$(P'_1P'_2P'_3P'_4)\sim(P'_{-3}P'_{-4}P'_{-1}P'_{-2}) \quad\text{or}\quad (P'_1P'_2P'_3P'_4)\sim(P'_{-4}P'_{-3}P'_{-2}P'_{-1}),$$

we can determine a correspondence between them such that corresponding pairs of lines intersect on $l_{\gamma\delta}$. We find that the lines $P'_1P'_3$, $P'_2P'_4$, $P'_{-1}P'_{-3}$, $P'_{-2}P'_{-4}$ intersect in P_γ, the lines $P'_1P'_4$, $P'_2P'_3$, $P'_{-1}P'_{-4}$, $P'_{-2}P'_{-3}$, in P_δ, the lines $P'_1P'_2$, $P'_{-3}P'_{-4}$, in P_ξ, and the lines $P'_3P'_4$, $P'_{-1}P'_{-2}$, in $P_{\xi'}$, where

$$(14) \qquad \begin{aligned} \xi &= p_{21}\nu\gamma+p_{12}\mu t\delta, & \xi' &= p_{21}\nu\gamma-p_{12}\mu t\delta,\\ \nu &= 3p_{12}^2t^2+p_{21}^2, & \mu &= p_{12}^2t^2+3p_{21}^2. \end{aligned}$$

It follows that in two ways these two quadrangles are in perspective from a point. Indeed the lines $P'_1P'_{-3}$, $P'_2P'_{-4}$, $P'_3P'_{-1}$, $P'_4P'_{-2}$ pass through P_α, and the lines $P'_1P'_{-4}$, $P'_2P'_{-3}$, $P'_3P'_{-2}$, $P'_4P'_{-1}$ pass through P_β.

If on the other hand we regroup the points into the sets $(P'_1P'_2P'_{-3}P'_{-4})$, $(P'_{-1}P'_{-2}P'_3P'_4)$, we find that these two sets lie in the respective planes

$$\begin{aligned} p_{12}\mu_1 t_1(x_1+x_3)-p_{21}\nu_1(x_2+x_4) &= 0,\\ p_{12}\mu_1 t_1(x_1+x_3)+p_{21}\nu_1(x_2+x_4) &= 0, \end{aligned}$$

on the first cone ray $l_{\alpha\beta}$. Moreover these two quadrangles are in perspective from the two points P_γ, P_δ, the four lines $P_1'P_3'$, $P_2'P_4'$, $P_{-3}'P_{-1}'$, $P_{-4}'P_{-2}'$ passing through P_γ, and the four lines $P_1'P_4'$, $P_2'P_3'$, $P_{-3}'P_{-2}'$, $P_{-4}'P_{-1}'$ passing through P_δ.

The equations of the planes osculating the four primary cone cubics at a set of corresponding points P_1, P_2, P_3, P_4 are, respectively,

$$
(15)\qquad
\begin{aligned}
(\pi_1)\quad & 3p_{12}^2 t^2 x_1 - p_{12}^2 t^3 x_2 - p_{21}^2 x_3 + 3p_{21}^2 t\, x_4 = 0,\\
(\pi_2)\quad & p_{21}^2 x_1 - 3p_{21}^2 t\, x_2 - 3p_{12}^2 t^2 x_3 + p_{12}^2 t^3 x_4 = 0,\\
(\pi_3)\quad & p_{21}^2 x_1 + p_{12}^2 t^3 x_2 - 3p_{12}^2 t^2 x_3 - 3p_{21}^2 t\, x_4 = 0,\\
(\pi_4)\quad & 3p_{12}^2 t^2 x_1 + 3p_{21}^2 t\, x_2 - p_{21}^2 x_3 - p_{12}^2 t^3 x_4 = 0.
\end{aligned}
$$

These four planes pass through a common point P_u on $l_{\gamma\delta}$, where

$$(16)\qquad u = \mu t\gamma + \nu\delta.$$

The equations of the planes osculating the four secondary cone cubics at the points P_1', P_2', P_3', P_4' are, respectively,

$$
(17)\qquad
\begin{aligned}
(\pi_1')\quad & p_{12}^3 t^3 x_1 - p_{12}^2 p_{21} t^2 x_2 + p_{12} p_{21}^2 t\, x_3 - p_{21}^3 x_4 = 0,\\
(\pi_2')\quad & p_{12} p_{21}^2 t\, x_1 - p_{21}^3 x_2 + p_{12}^3 t^3 x_3 - p_{12}^2 p_{21} t^2 x_4 = 0,\\
(\pi_3')\quad & p_{12} p_{21}^2 t\, x_1 + p_{12}^2 p_{21} t^2 x_2 + p_{12}^3 t^3 x_3 + p_{21}^3 x_4 = 0,\\
(\pi_4')\quad & p_{12}^3 t^3 x_1 + p_{21}^3 x_2 + p_{12} p_{21}^2 t\, x_3 + p_{12}^2 p_{21} t^2 x_4 = 0.
\end{aligned}
$$

These four planes pass through a common point P_v on $l_{\alpha\beta}$, where

$$(18)\qquad v = p_{21}\alpha + p_{12} t\beta.$$

The common tangent planes to Q and Q' at the points P_α, P_β are given by the equations

$$(19)\qquad (c)\quad x_2 + x_4 = 0, \qquad (d)\quad x_1 + x_3 = 0.$$

From (17) and (19) we discover that the following sets of planes are collinear:

$$(\pi_1', \pi_3', d),\quad (\pi_2', \pi_4', d),\quad (\pi_1', \pi_4', c),\quad (\pi_2', \pi_3', c).$$

The plane determined by the line of intersection of π_1', π_2' and the line $l_{\alpha\beta}$ has for its equation

$$(20)\qquad p_{12} t(x_1 + x_3) - p_{21}(x_2 + x_4) = 0,$$

and the plane determined by the line of intersection of π_3', π_4' and the line $l_{\alpha\beta}$ has for its equation

$$(21) \qquad p_{12}\, t(x_1 + x_3) + p_{21}(x_2 + x_4) = 0.$$

The four planes (c), (d), (20), (21) are all on $l_{\alpha\beta}$, the second pair being harmonic conjugates with respect to the first pair, and vice versa. We may write the equations of the planes osculating the four secondary cone cubics at the points P'_{-1}, P'_{-2}, P'_{-3}, P'_{-4} by replacing t with $-t$ in (17). These new planes will pass through the point

$$v' = p_{21}\,\alpha - p_{12}\,t\boldsymbol{\beta},$$

on $l_{\alpha\beta}$, and for this set of planes the following triples are collinear:

$$(\pi'_{-1},\ \pi'_{-3},\ d),\quad (\pi'_{-2},\ \pi'_{-4},\ d),\quad (\pi'_{-1},\ \pi'_{-4},\ c),\quad (\pi'_{-2},\ \pi'_{-3},\ c).$$

The plane determined by the line of intersection of π'_{-1}, π'_{-2} and $l_{\alpha\beta}$ is precisely that given by (21), while the plane determined by the line of intersection of π'_{-3}, π'_{-4} and $l_{\alpha\beta}$ is given by (20). It follows that in either of two orders,

$$(\pi_1'\,\pi_2'\,\pi_3'\,\pi_4') \sim (\pi'_{-3}\,\pi'_{-4}\,\pi'_{-1}\,\pi'_{-2}) \quad \text{or} \quad (\pi_1'\,\pi_2'\,\pi_3'\,\pi_4') \sim (\pi'_{-4}\,\pi'_{-3}\,\pi'_{-2}\,\pi'_{-1}),$$

we can determine a correspondence between these two sets of four planes such that corresponding pairs of lines determine planes on the line $l_{\alpha\beta}$. We find in fact that the lines $\pi_1'\pi_3'$, $\pi_2'\pi_4'$, $\pi'_{-1}\pi'_{-3}$, $\pi'_{-2}\pi'_{-4}$ lie on plane d, the lines $\pi_1'\pi_4'$, $\pi_2'\pi_3'$, $\pi'_{-1}\pi'_{-4}$, π'_{-2}, π'_{-3} lie on plane c, the lines $\pi_1'\pi_2'$, $\pi'_{-3}\pi'_{-4}$ lie on plane (20), and the lines $\pi_3'\pi_4'$, $\pi'_{-1}\pi'_{-2}$ lie on plane (21).

It results that in either of two ways these two sets of four planes are in perspective from a plane. It can be verified that the four lines $\pi_1'\pi'_{-3}$, $\pi_2'\pi'_{-4}$, $\pi_3'\pi'_{-1}$, $\pi_4'\pi'_{-2}$ are coplanar, as are also the four lines $\pi_1'\pi'_{-4}$, $\pi_2'\pi'_{-3}$, $\pi_3'\pi'_{-2}$, $\pi_4'\pi'_{-1}$, the two planes having for their respective equations

$$(22) \qquad (b)\quad x_1 - x_3 = 0, \qquad\qquad (a)\quad x_2 - x_4 = 0.$$

Planes (a), (b) are common tangent planes to Q and Q' at the respective points P_γ, P_δ, on $l_{\gamma\delta}$.

Without further discussion we note that in either of two ways the two sets of planes $(\pi_1'\,\pi_2'\,\pi'_{-3}\,\pi'_{-4})$, $(\pi_3'\,\pi_4'\,\pi'_{-1}\,\pi'_{-2})$ are in perspective from a plane, the four lines $\pi_1'\pi_3'$, $\pi_2'\pi_4'$, $\pi'_{-3}\pi'_{-1}$, $\pi'_{-4}\pi'_{-2}$ lying on the plane (d) and the four lines $\pi_1'\pi_4'$, $\pi_2'\pi_3'$, $\pi'_{-3}\pi'_{-2}$, $\pi'_{-4}\pi'_{-1}$ lying

on the plane (c). Similar perspectivities exist between the two sets of planes π_1, π_2, π_3, π_4, and π_{-1}, π_{-2}, π_{-3}, π_{-4}. It is not necessary to dwell further upon this.

We close this part of our discussion by emphasizing the duality which exists between the primary cone cubics, thought of as point loci, and the secondary cone cubics, thought of as the loci of their osculating planes, this duality being of a reciprocal nature.

IV. Other properties of the cone cubics. Allied curves

The perspectivities of the cone cubics are by no means their only interesting properties. Without going into unnecessary detail, we will establish in this section a number of theorems which will serve to illustrate the wealth of material awaiting further investigation.

Each of the primary cone cubics determines a linear complex*. Since $C_1, \cdots, C_4$ are projectively equivalent to the primary flecnode cubic C_F, the corresponding four linear complexes will be projectively equivalent to to the complex L_1. We obtain the equations, in line coördinates, of these four complexes by applying to (L_1) of (4) the transformations of line coördinates which are the consequences of the four transformations (5_1), (5_2), (5_3), (5_4) in point coördinates. We write below, for each of these complexes, its equation in line coördinates together with the point-plane correspondence which it determines. We have

$$
(23)\quad
\begin{aligned}
&(L_{11})\ \ 3\omega_{14}-\omega_{23}=0,\ u_1=3x_4,\ u_2=-x_3,\ u_3=x_2,\ u_4=-3x_1;\\
&(L_{12})\ \ \omega_{14}-3\omega_{23}=0,\ u_1=x_4,\ u_2=-3x_3,\ u_3=3x_2,\ u_4=-x_1;\\
&(L_{13})\ \ \omega_{12}+3\omega_{34}=0,\ u_1=x_2,\ u_2=-x_1,\ u_3=3x_4,\ u_4=-3x_3;\\
&(L_{14})\ \ 3\omega_{12}+\omega_{34}=0,\ u_1=3x_2,\ u_2=-3x_1,\ u_3=x_4,\ u_4=-x_3.
\end{aligned}
$$

The points $A_1, \cdots, A_4$ which correspond to plane (10) by means of these four linear complexes have for their coördinates, according to (23),

$$
(24)\qquad
\begin{array}{lcccc}
 & A_1 & A_2 & A_3 & A_4 \\
x_1 = & t, & 3t, & 3t, & t, \\
x_2 = & 3, & 1, & 3, & 1, \\
x_3 = & 3t, & t, & t, & 3t, \\
x_4 = & 1, & 3, & 1, & 3.
\end{array}
$$

The loci $a_1, \cdots, a_4$ of these four points are of course straight lines, the polar reciprocals of $l_{\alpha\beta}$ with respect to the four complexes $L_{11}, \cdots, L_{14}$.

* The four linear complexes determined by the secondary cone cubics are identical with those determined by the primary cone cubics.

From (24) and (2) we find that the quadrangles $P_1P_2P_3P_4$ and $A_1A_2A_3A_4$, both lying in plane (10), are in perspective from the point P; $(t, 1, t, 1)$, in which this plane is cut by $l_{\gamma\delta}$. In brief, *the polar reciprocals of the first cone ray, taken with respect to the linear complexes determined by the primary cone cubics, determine on each plane of the primary pencil associated with g a quadrangle which is in perspective with the quadrangle of the primary cone cubic points of this plane, the locus of the center of perspective being the second cone ray.*

A number of similar theorems may be readily obtained by interchanging cone rays and by making use of the secondary, rather than the primary, cone cubic points. We leave these to be enunciated by the reader.

We have already seen that the equation of the general plane on the second cone ray is

$$x_1 - x_3 - t(x_2 - x_4) = 0. \tag{25}$$

The points $B_1, \cdots, B_4$ which correspond to the plane (25) by means of the four complexes $L_{11}, \cdots, L_{14}$ and whose loci $b_1, \cdots, b_4$ are the polar reciprocals of $l_{\gamma\delta}$, have for their coördinates, by (23),

$$\begin{array}{rrrrr} & B_1 & B_2 & B_3 & B_4 \\ x_1 = & t, & 3t, & 3t, & t, \\ x_2 = & 3, & 1, & 3, & 1, \\ x_3 = & -3t, & -t, & -t, & -3t, \\ x_4 = & -1, & -3, & -1, & -3. \end{array} \tag{26}$$

From (24), (26), and (1) we find that, of the eight lines involved, a_1, a_2, b_1, b_2 lie on Q' and a_3, a_4, b_3, b_4 lie on Q. Moreover the points $(a_1 a_4)$, $(a_2 a_3)$, $(b_1 b_4)$, $(b_2 b_3)$ are on the flecnode tangent $l_{y\varrho}$ and the points $(a_1 a_3)$, $(a_2 a_4)$, $(b_1 b_3)$, $(b_2 b_4)$ are on the flecnode tangent $l_{z\sigma}$. Summing up these results we find that *the four polar reciprocals of the first (second) cone ray, taken with respect to the linear complexes determined by the primary cone cubics associated with g, constitute four edges of a tetrahedron whose other two edges are the flecnode tangents, and of these four lines two lie upon each of the complex quadrics.*

The four planes (17) are in general distinct, but when $t = p_{21}/p_{12}$, the first and second coincide in the plane whose equation is

$$x_1 - x_2 + x_3 - x_4 = 0, \tag{27$_1$}$$

and the third and fourth coincide in the plane whose equation is

$$x_1 + x_2 + x_3 + x_4 = 0. \tag{27$_2$}$$

When $t = -p_{21}/p_{12}$, the first pair coincide in plane (27_2) and the second pair in plane (27_1). If we take $t = \pm p_{21}\, i/p_{12}$, the same situation again develops, but this time our planes of coincidence have the equations

$$(27_3)\qquad x_1 + ix_2 - x_3 - ix_4 = 0,$$

$$(27_4)\qquad x_1 - ix_2 - x_3 + ix_4 = 0.$$

From (8) we see that the two real planes of this set are on $l_{\alpha\beta}$ and the two imaginary planes are on $l_{\gamma\delta}$. It is interesting to note also that these planes are given by those values of t which give the focal points of the primary cone cubics. From the above considerations it follows that *the secondary cone cubics have four osculating planes in common, a real pair intersecting in the first cone ray and an imaginary pair intersecting in the second cone ray. Moreover in each of these planes lie two pairs of secondary cone cubic points. These pairs correspond to two of the four focal points of the primary cone cubics.* The four planes (27) may be called the *focal planes of the secondary cone cubics.*

From (17) and (1) we see that the planes π_1', π_2' are tangent to Q and the planes π_3', π_4' are tangent to Q'. The points of contact have for their coördinates

$$(28)\quad \begin{array}{lllll}
(C_1'') & x_1 = p_{21}^3, & x_2 = p_{12}p_{21}^2 t, & x_3 = -p_{12}^2 p_{21} t^2, & x_4 = -\ \ p_{12}^3 t^3; \\
(C_2'') & x_1 = p_{12}^2 p_{21} t^2, & x_2 = \ \ p_{12}^3 t^3, & x_3 = -p_{21}^3, & x_4 = -p_{12}p_{21}^2 t; \\
(C_3'') & x_1 = p_{12}^2 p_{21} t^2, & x_2 = p_{12}p_{21}^2 t, & x_3 = -p_{21}^3, & x_4 = -\ \ p_{12}^3 t^3; \\
(C_4'') & x_1 = p_{21}^3, & x_2 = \ \ p_{12}^3 t^3, & x_3 = -p_{12}^2 p_{21} t^2, & x_4 = -p_{12}p_{21}^2 t.
\end{array}$$

As t varies these four points trace cubics $C_1'', \cdots, C_4''$, two lying upon Q and two upon Q'. We shall call these four curves *primary contact cubics.*

The planes osculating the four curves $C_1'', \cdots, C_4''$ are given by the respective equations

$$(29)\quad \begin{array}{r}
p_{12}^3 t^3 x_1 - 3p_{12}^2 p_{21} t^2 x_2 - 3p_{12}p_{21}^2 t\, x_3 + \qquad p_{21}^3\ \ x_4 = 0, \\
3p_{12}p_{21}^2 t\, x_1 - \qquad p_{21}^3\ \ x_2 - \qquad p_{12}^3 t^3 x_3 + 3p_{12}^2 p_{21} t^2 x_4 = 0, \\
3p_{12}p_{21}^2 t\, x_1 - 3p_{12}^2 p_{21} t^2 x_2 - \qquad p_{12}^3 t^3 x_3 + \qquad p_{21}^3\ \ x_4 = 0, \\
p_{12}^3 t^3 x_1 - \qquad p_{21}^3\ \ x_2 - 3p_{12}p_{21}^2 t\, x_3 + 3p_{12}^2 p_{21} t^2 x_4 = 0,
\end{array}$$

and the points which correspond to these planes by means of L are, respectively,

$$
(30)\quad
\begin{array}{lllll}
(C_1''') & x_1 = 3p_{21}^2 t, & x_2 = p_{21}^2, & x_3 = p_{12}^2 t^3, & x_4 = 3p_{12}^2 t^2; \\
(C_2''') & x_1 = p_{12}^2 t^3, & x_2 = 3p_{12}^2 t^2, & x_3 = 3p_{21}^2 t, & x_4 = p_{21}^2; \\
(C_3''') & x_1 = p_{12}^2 t^3, & x_2 = p_{21}^2, & x_3 = 3p_{21}^2 t, & x_4 = 3p_{12}^2 t^2; \\
(C_4''') & x_1 = 3p_{21}^2 t, & x_2 = 3p_{12}^2 t^2, & x_3 = p_{12}^2 t^3, & x_4 = p_{21}^2.
\end{array}
$$

Equations (30) define four new cubics $C_1''', \cdots, C_4'''$, which, by virtue of their relation to the primary contact cubics, we shall speak of as *secondary contact cubics.*

The equations of the osculating planes of the curves $C_1''', \cdots, C_4'''$ are, respectively,

$$
(31)\quad
\begin{aligned}
p_{12}^2 t^2 x_1 - p_{12}^2 t^3 x_2 + p_{21}^2 x_3 - p_{21}^2 t\, x_4 &= 0, \\
p_{21}^2 x_1 - p_{21}^2 t\, x_2 + p_{12}^2 t^2 x_3 - p_{12}^2 t^3 x_4 &= 0, \\
p_{21}^2 x_1 - p_{12}^2 t^3 x_2 + p_{12}^2 t^2 x_3 - p_{21}^2 t\, x_4 &= 0, \\
p_{12}^2 t^2 x_1 - p_{21}^2 t\, x_2 + p_{21}^2 x_3 - p_{12}^2 t^3 x_4 &= 0.
\end{aligned}
$$

A comparison of equations (31) with equations (1) and (2) shows that these planes are tangent to the complex quadrics, the locus of the points of contact being, for the first two, the cubics C_1, C_2 on Q, and for the second two the cubics C_3, C_4 on Q'. Starting with the four primary cone cubics we have thus, after four point transformations of space, returned to these same cubics, and have in the process introduced three other sets of four curves each, all of them cubics. Let us further examine this closed sequence of transformations.

To each point of space there corresponds by means of L a plane, and to this plane there corresponds by means of L_{11} a point. These two complexes thus determine a point transformation whose analytic expression we proceed to find. The point-plane correspondence determined by L* is given by

$$
(32)\quad u_1 = p_{12} x_3, \quad u_2 = -p_{21} x_4, \quad u_3 = -p_{12} x_1, \quad u_4 = p_{21} x_2.
$$

From (32) and the first of equations (23) the equations of this point transformation are easily obtained. They are

$$
\bar{x}_1 = -p_{21} x_2, \quad \bar{x}_2 = -3p_{12} x_1, \quad \bar{x}_3 = 3p_{21} x_4, \quad \bar{x}_4 = p_{12} x_3.
$$

Associating thus with L each of the complexes of (23) in turn, we obtain four such point transformations. They are

* Proj. Dif. Geom., p. 206.

$$(33)\quad \begin{aligned} &(L, L_{11})\ \bar{x}_1 = -\ p_{21}x_2,\ \bar{x}_2 = -3p_{12}x_1,\ \bar{x}_3 = 3p_{21}x_4,\ \bar{x}_4 = \ p_{12}x_3;\\ &(L, L_{12})\ \bar{x}_1 = -3p_{21}x_2,\ \bar{x}_2 = -\ p_{12}x_1,\ \bar{x}_3 = \ p_{21}x_4,\ \bar{x}_4 = 3p_{12}x_3;\\ &(L, L_{13})\ \bar{x}_1 = -3p_{21}x_4,\ \bar{x}_2 = -3p_{12}x_3,\ \bar{x}_3 = \ p_{21}x_2,\ \bar{x}_4 = \ p_{12}x_1;\\ &(L, L_{14})\ \bar{x}_1 = -\ p_{21}x_4,\ \bar{x}_2 = -\ p_{12}x_3,\ \bar{x}_3 = 3p_{21}x_2,\ \bar{x}_4 = 3p_{12}x_1. \end{aligned}$$

We note in passing that these transformations are each of period two and hence that the point correspondences determined by them are reciprocal.

Since each quadric determines a (1,1) correspondence between the points and planes of space, we may set up a point transformation by making use of a linear complex and a quadric. For to each point there corresponds its polar plane by means of the complex and to this plane there corresponds its pole with respect to the quadric. Making use of this notion we define a second set of four point transformations, their expressions being

$$(34)\quad \begin{aligned} &(L_{11}, Q)\quad \bar{x}_1 = -3x_1,\ \bar{x}_2 = -\ x_2,\ \bar{x}_3 = \ x_3,\ \bar{x}_4 = \ 3x_4;\\ &(L_{12}, Q)\quad \bar{x}_1 = -\ x_1,\ \bar{x}_2 = -3x_2,\ \bar{x}_3 = 3x_3,\ \bar{x}_4 = \ x_4;\\ &(L_{13}, Q')\quad \bar{x}_1 = -\ x_1,\ \bar{x}_2 = \ x_2,\ \bar{x}_3 = 3x_3,\ \bar{x}_4 = -3x_4;\\ &(L_{14}, Q')\quad \bar{x}_1 = -3x_1,\ \bar{x}_2 = \ 3x_2,\ \bar{x}_3 = \ x_3,\ \bar{x}_4 = -\ x_4. \end{aligned}$$

The eight transformations of (33) and (34), taken four at a time in the proper order, carry the four primary cone cubics through their four-phase cycle. Symbolically we have

$$(35)\quad \begin{aligned} &(L_{12}, Q)\ \big[(L, L_{12})\,[(L_{11}, Q)\ [(L, L_{11})\,C_1 = C_1'] = C_1''] = C_1'''\big] = C_1,\\ &(L_{11}, Q)\ \big[(L, L_{11})\,[(L_{12}, Q)\ [(L, L_{12})\,C_2 = C_2'] = C_2''] = C_2'''\big] = C_2,\\ &(L_{14}, Q')\,\big[(L, L_{14})\,[(L_{13}, Q')\,[(L, L_{13})\,C_3 = C_3'] = C_3''] = C_3'''\big] = C_3,\\ &(L_{13}, Q')\,\big[(L, L_{13})\,[(L_{14}, Q')\,[(L, L_{14})\,C_4 = C_4'] = C_4''] = C_4'''\big] = C_4. \end{aligned}$$

It has been noted that the primary and secondary cone cubics determine the same set $L_{11}, L_{12}, L_{13}, L_{14}$ of linear complexes. Without difficulty it can be shown that the primary and secondary contact cubics determine the same four linear complexes, but in the order $L_{12}, L_{11}, L_{14}, L_{13}$. Many additional properties of these curves might be developed. It will be sufficient however to summarize the results of the last few paragraphs and then to state without proof a number of additional theorems whose truth can be demonstrated with the material at hand.

We find that *associated with each line element g of the general ruled surface there are sixteen projectively equivalent space cubics so related in cyclically ordered sets of four each that the points of the curves in any set lie upon the osculating planes of the curves of the preceding set.*

The curves of two of the four sets lie upon the complex quadrics associated with g, two from each set on each quadric, while the osculating planes of the curves of the other two sets are tangent to these quadrics.

The four linear complexes determined by the curves of any set are distinct, but any two sets determine the same four complexes.

Among the theorems whose proofs are left to the reader we have the following:

1. The primary (secondary) contact cubics are the loci of the poles of the osculating planes of the secondary (primary) cone cubics taken with respect to the complex quadrics, and conversely.

2. The cubics C_j^k $(j=1,\cdots,4;\ k=0,\cdots,3)$ belong to the complexes L_{1i}, C_j and C_j' belonging to L_{1i} $(i=j)$, and C_j'' and C_j''' belonging to L_{1i} where $i=j-(-1)^j$.

3. The ruled surfaces determined by the point correspondences set up between the pairs of cubics C_j^k, C_j^{k+1} $(j=1,\cdots,4;\ k=0,\cdots,3)$, by the parameter t, belong to one of the four complexes L_{1i}, those determined by C_j, C_j' and C_j', C_j'' belonging to $L_{1i}\,(i=j)$, and those determined by C_j'', C_j''' and C_j''', C_j belonging to L_{1i} where $i=j-(-1)^j$.

4. Each of the cubics C_j^k generates a surface as the line element g with which it is associated varies over the ruled surface S. Of these surfaces the eight generated by the primary cone cubics and the secondary contact cubics are projectively equivalent, as are also the eight generated by the secondary cone cubics and the primary contact cubics.

5. Four of the sixteen surfaces S_j^k generated by the cubics C_j^k are tangent to the ruled surface S, S_1' and S_1'' being tangent to S along the branch C_y of the flecnode curve, and S_1 and S_1''' tangent to S along the branch C_z of this curve. Of the remaining twelve surfaces, S_2 and S_2''' cut S along C_y, S_2' and S_2'' cut S along C_z, while S_3, S_4', S_4'', and S_3''' cut S along both C_y and C_z.

6. The point correspondence existing between each pair of primary cone cubics C_j determines a ruled surface on which this pair of cubics are directrix curves. Of the six surfaces thus determined, those given by C_1, C_2 and by C_3, C_4 are the two complex quadrics. The remaining four are cubic cones with vertices at the points P_α, P_β. Their equations, in the system of coördinates here employed, are

$$
\begin{array}{lll}
(C_1, C_3) & p_{12}^2 x_2 (x_1+x_3)^2 + p_{21}^2 x_4 (x_2+x_4)^2 = 0, & \text{vertex at } P_\alpha; \\
(C_1, C_4) & p_{12}^2 x_1 (x_1+x_3)^2 + p_{21}^2 x_3 (x_2+x_4)^2 = 0, & \quad " \quad " \quad P_\beta; \\
(C_2, C_3) & p_{12}^2 x_3 (x_1+x_3)^2 + p_{21}^2 x_1 (x_2+x_4)^2 = 0, & \quad " \quad " \quad P_\beta; \\
(C_2, C_4) & p_{12}^2 x_4 (x_1+x_3)^2 + p_{21}^2 x_2 (x_2+x_4)^2 = 0, & \quad " \quad " \quad P_\alpha.
\end{array}
$$

7. The primary contact cubics determine among themselves six ruled surfaces two of which are the complex quadrics, and the remaining four, cubic cones. The equations of the latter four are

$$
\begin{array}{lll}
(C_1'', C_3'') & x_4(x_1-x_3)^2+x_2(x_2-x_4)^2=0, & \text{vertex at } P_\gamma; \\
(C_1'', C_4'') & x_3(x_1-x_3)^2+x_1(x_2-x_4)^2=0, & \quad ,, \quad ,, \; P_\delta; \\
(C_2'', C_3'') & x_1(x_1-x_3)^2+x_3(x_2-x_4)^2=0, & \quad ,, \quad ,, \; P_\delta; \\
(C_2'', C_4'') & x_2(x_1-x_3)^2+x_4(x_2-x_4)^2=0, & \quad ,, \quad ,, \; P_\gamma.
\end{array}
$$

No attempt has been made in this paper to investigate the properties of the loci which the points, lines, curves and surfaces here discussed will generate when the line-element g with which they are associated varies over the surface S. Nor has it been thought advisable to consider the results of imposing upon S any special conditions. The methods of attacking all of these problems are available and their solutions, while requiring some ingenuity, should involve no great difficulties.

University of Washington,
Seattle, Wash.

CONCERNING UPPER SEMI-CONTINUOUS COLLECTIONS OF CONTINUA*

BY

R. L. MOORE

I. Upper semi-continuous collections of continua which do not separate a plane

A collection of continua is said to be an *upper semi-continuous* collection if for each element g of the collection G and each positive number e there exists a positive number d such that if x is any element of G at a lower distance† from g less than d then the upper distance of x from g is less than e. The element p of such a collection G is said to be a *limit element* of the subcollection K of G if for every positive number e there exists some element of K which is distinct from p and whose upper distance from p is less than e.

In this section it will be shown that if, in a plane S, G is any upper semi-continuous collection of mutually exclusive bounded continua such that every point of S belongs to some continuum of the collection G and no continuum of G separates S, then if each continuum of G is considered as a point, and the term region is suitably defined, all the Axioms 1–8 of the author's article‡ *On the foundations of plane analysis situs* hold true, if the space S of that article is interpreted to mean the collection of elements G. Thus the set of elements G is, with respect to the notion of limit point defined in that paper, *topologically equivalent to the set of ordinary points in a plane S.* Furthermore the notion of limit point so defined coincides, for the case of an upper semi-continuous collection, with the natural interpretation of limit element given above.

From here on, in this section, it is understood that there has been selected some definite upper semi-continuous collection of bounded continua

* Presented to the Society, December 30, 1924.

† If M is a point set and P is a point, then by $l(PM)$ is meant the lower bound of the distances from P to all the different points of M. If M and N are two point sets, then by $l(MN)$ is meant the lower bound of the values $[l(PN)]$ for all points P of M, while by $u(MN)$ is meant the upper bound of these values for all points P of M. It is to be observed that $u(MN)$ may be different from $u(NM)$. The point set M is said to be at the upper distance $u(MN)$ from the point set N and is said to be at the lower distance $l(MN)$ from N. According to this terminology N is not always at the upper distance $u(MN)$ from M.

‡ These Transactions, vol. 17 (1916), pp. 131–164. Hereafter this paper will be referred to as F. A.

such that no one of these continua separates S and such that every point of S belongs to just one of them, and the letter G will be used throughout to denote this particular upper semi-continuous collection.

DEFINITIONS. A set of elements of the set G is said to be the *sum* of the sets K_1 and K_2 if every element which belongs to K_1 or to K_2 belongs also to K and every element which belongs to K belongs either to K_1 or to K_2. Two sets of elements of G are said to be *mutually exclusive* if there is no element of G which belongs to both of them. Two sets of elements of G are said to be *mutually separated* if they are mutually exclusive and neither of them contains a limit element of the other one. A set of elements G is said to be connected if it is not the sum of two mutually separated sets of elements of G. A set of elements of G is said to be *closed* if it contains all its limit elements. A set of elements of G is said to be a *continuum* of elements of G if it is both closed and connected. A set K of elements of G is said to be *bounded* if the point set obtained by adding together the points of all the elements of G is a bounded set of points

DEFINITION. A bounded subset H of the set G is said to be a *simple closed curve* (of elements of G) provided it is disconnected by the omission of any two of its elements.

DEFINITION. If A and B are two distinct elements of G, and H is a closed, connected and bounded set of elements of G, and A and B belong to H, and H is disconnected by the omission of any one of its elements except A and B, then H is said to be a *simple continuous arc* (of elements of G) from A to B, and A and B are said to be the *extremities*, or the *end-elements*, of this arc.

DEFINITION. A *domain* of elements of G is a connected set D of elements of G such that for every element x belonging to D there exists a positive number d such that if y is any element of G at an upper distance from x less than d then y belongs to D.

DEFINITION. By the *boundary* of a set H of elements of G is meant the set of all elements $[x]$ of H such that x either belongs to H and is a limit element of the set $G-H$ or x does not belong to H but it is a limit element of H.

DEFINITION. A domain of elements of G is said to be a *complementary* domain of a closed set H of elements of G provided the boundary of D is a subset of H.

DEFINITION. The *outer* boundary of a bounded domain D of elements of G is the boundary of the unbounded complementary domain of the boundary of D.

NOTATION. In this paper if a letter is used to denote a set of elements belonging to G then the same letter with a bar above it will denote the

set of *points* obtained by adding together the points of all the elements of that set of elements.

THEOREM 1. *If K is a set of points and H is the set of all elements $[g]$ of the collection G such that g contains at least one point of K, then H is closed if K is closed and H is connected if K is connected.*

Proof. I. Suppose that K is closed. Suppose that p is a limit element of H. Then for every positive integer n there exists an element p_n belonging to H and such that each point of p_n is at a distance less than $1/n$ from some point of p, and such that if $i \neq j$ then p_i is distinct from p_j. But each p_n contains a point P_n belonging to K. For each n there exists in p a point X_n such that the distance from P_n to X_n is less than $1/n$. Since p is closed and bounded the sequence $X_1, X_2, \cdots$ contains a subsequence of distinct points which has as its sequential limit point some point X in p. The point X is a limit point of $P_1, P_2, \cdots$. Since every P_n belongs to K and K is closed, therefore X belongs to K. Thus p contains a point of K and therefore p belongs to H. Thus H contains all of its limit elements. In other words it is closed.

II. Suppose that K is connected. Then H is connected. For suppose, on the contrary, that H is the sum of two mutually separated sets of elements H_1 and H_2. Let K_1 denote the set of points common to K and $\overline{H}_1$ and let K_2 denote the set common to K and $\overline{H}_2$. Since K is connected, either K_1 contains a limit point of K_2 or K_2 contains a limit point of K_1. Suppose that K_1 contains a point P which is a limit point of K_2. Let p denote that element of H_1 which contains p. Since G is an upper semi-continuous collection, if e is a positive number there exists a positive number d such that if an element of H contains one point whose distance from P is less than d then *every* point of that element is at a distance less than e from some point of p. But, since P is a limit point of K_2, K_2 contains a point P_d at a distance from P less than d. Hence if h denotes that element of H_2 which contains P_d then every point of h is at a distance less than e from some point of p. Thus p is a limit element of H_2, contrary to the supposition that H_1 and H_2 are mutually separated. A similar contradiction would be obtained if it were supposed that K_2 contains a limit point of K_1. Thus the supposition that H is not connected leads to a contradiction.

THEOREM 2. *If D is a bounded complementary domain of a bounded continuum of elements of G, and K is the outer boundary of D, and p is an element of K, then K is a continuum of elements of G and $K-p$ is connected.*

Proof. Let E denote the unbounded complementary domain of the boundary of D and let B denote the set of *points* which constitutes the boundary of $\overline{E}$. Since each point of B belongs to some element of K and each element of K contains a point of B and B is a closed and

connected set of points, therefore, by Theorem 1, K is a closed and connected set of elements of G. I will proceed to show that if p is an element of K then $K-p$ is connected. Suppose, on the contrary, that $K-p$ is the sum of two mutually separated sets of elements H and N. Then clearly $H+p$ and $N+p$ are both closed and connected sets of elements and they have in common only the element p. Let x and y denote elements belonging to D and E respectively. Let X and Y denote points belonging to x and y respectively. Since the continua $\bar{N}+\bar{p}$ and $\bar{H}+\bar{p}$ have in common only the continuum $\bar{p}$, and $\bar{N}+\bar{p}+(\bar{H}+\bar{p})$ separates X from Y, therefore* either $\bar{N}+\bar{p}$ or $\bar{H}+\bar{p}$ separates X from Y. Suppose that $\bar{H}+\bar{p}$ does. Then $H+p$ separates x from y, that is to say $G-(H+p)$ is the sum of two mutually separated sets of elements of G such that one of these sets contains x and the other one contains y. Let D_x and D_y denote the complementary domains of $H+p$ that contain x and y respectively. Clearly D_x contains D. Let q denote an element of G that belongs to the set N. The element q is a limit element of D and therefore of D_x. But q does not belong to the boundary of D_x. Hence it belongs to D_x. But q is also a limit element of E. Thus D_x contains an element of E and therefore, since E is connected and contains no element of the boundary of D_x, E is a subset of D_x. Thus y belongs to D_x, contrary to supposition. Similarly the supposition that $\bar{N}+\bar{p}$ separates X from Y would lead to a contradiction. The truth of Theorem 2 is therefore established.

DEFINITION. A *region* (of elements of G) is a bounded domain (of elements of G) which has a connected boundary.

THEOREM 3. *If p is an element of G and e is a positive number there exists a region of elements of G such that every element of this region is at an upper distance less than e from the element p.*

Proof. Since the set of points p does not separate S there exists† a simple closed curve (of ordinary *points*) J enclosing p and such that every point on or within J is at a distance less than e from some point of p. Let H denote the set of all elements $[x]$ of G such that the point set x contains at least one point of J. The point set $\bar{H}$ is a continuum. Let D denote that complementary domain of $\bar{H}$ which contains the point set p and let B denote the boundary of D. Let R denote the set of all elements $[g]$ of G such that the point set g is a subset of D. By a theorem of Brouwer's,‡ B is a closed and connected set of points. But

* Cf. S. Janiszewski, *Sur les coupures du plan faites par les continus*, Prace Matematyczno-Fizyczne, vol. 26 (1913).

† See Theorem 1 of my paper *Concerning the separation of point sets by curves*, Proceedings of the National Academy of Sciences, vol. 11 (1925), pp. 469–476.

‡ L. E. J. Brouwer, Mathematische Annalen, vol. 69.

the boundary of R consists of all those elements $[g]$ of G such that g contains a point of B. It follows that the boundary of R is connected. But clearly R is a domain. Hence it is a region. Every element which belongs to it is at an upper distance less than e from the element p.

THEOREM 4. *If p is an element of G and K is a set of elements of G then p is a limit element of K if and only if every region (of elements of G) which contains p contains at least one element of K which is distinct from p.*

The truth of Theorem 4 may be easily established with the help of Theorem 3.

By methods largely analogous to those used in a similar connection in my paper *Concerning the prime parts of a continuum** it may be shown that if the word "point" as used in F. A. is interpreted to mean "element of G" (and thus the set of all "points" is identified with the set of elements of G) and the word "region" as used therein is interpreted to mean "region of elements of G" as defined above, then, for the space $S(G)$ consisting of all such "points" (elements of G), Axioms 1, 2, 4 and 5 and Theorem 4 of F. A. hold true and consequently Theorems 1–15 of that paper all hold true. For an indication of material sufficient for a proof, based on this result, of the following Theorems 5, 6 and 7, see page 139 of F. A. and page 342 of my paper *Concerning simple continuous curves.*†

THEOREM 5. *No simple continuous arc of elements of G is disconnected by the omission of either of its extremities.*

THEOREM 6. *If K is a simple continuous arc of elements of G, then every closed and connected subset of K which contains more than one element is itself a simple continuous arc of elements of G.*

THEOREM 7. *If pq is a simple continuous arc of elements of G with p and q as its extremities, x is an element of G belonging to the set pq, but distinct from p and q, and H is a set of elements of G belonging to pq, then p is a limit element of the set H if and only if every simple continuous arc of elements of G which contains x and is a subset of pq (but does not have x as an extremity) contains at least one element of the set H which is distinct from x.*

THEOREM 8. *If K is a closed, connected and bounded set of elements of G, and H is a connected proper subset of K, then the set $K-H$ contains an element of G whose omission does not disconnect K.*

Proof. Suppose, on the contrary, that K is disconnected by the omission of any element of $K-H$. Let p denote some definite element of $K-H$. Then $K-p$ is the sum of two mutually separated sets of elements. Since H is connected and does not contain p it must belong

* Mathematische Zeitschrift, vol. 22 (1925), pp. 307–315.

† These Transactions, vol. 21 (1920), pp. 333–347.

wholly to one of these sets. Let K_1 denote the one of which H is a subset and let K_2 denote the other one. It is easy to see that K_1+p and K_2+p are both closed and connected. The set K_2 contains some element whose omission does not disconnect K_2+p. For otherwise, if q denotes some element of K_2, K_2+p would be a simple continuous arc having p and q as its extremities and therefore, by Theorem 5, K_2+p would not be disconnected by the omission of q, contrary to supposition. Let r denote an element of K_2 whose omission does not disconnect K_2+p. Since K_2+p-r and K_1+p are connected sets having the element p in common their sum is connected. But their sum is $K-r$. Thus K is not disconnected by the omission of the element r. But since r belongs to K_2 it belongs to $K-H$.

With the help of Theorems 2 and 6 the following theorem may be established by an argument largely similar to that used in the proof of Lemma 8 of my paper *Concerning the prime parts of certain continua which separate the plane.**

THEOREM 9. *If pq is a simple continuous arc of elements of G, then $G-pq$ is connected.*

THEOREM 10. *If J is a simple closed curve of elements of G and p and q are distinct elements of J, then J is the sum of two simple continuous arcs (of elements of G) which have p and q as their extremities and which have in common no element except p and q.*

In view of results established above Theorem 10 is a consequence of Theorem 4 of my paper *Concerning simple continuous curves.*†

THEOREM 11. *If J is a simple closed curve of elements of G, then $G-J$ is the sum of two domains (of elements of G). Only one of these domains is bounded and J is the boundary of each of them.*

Proof. Let p and q denote two distinct elements of G belonging to the set J. By Theorem 10, J is the sum of two simple continuous arcs a and b which have p and q as their extremities but which have in common no other element of G. By Theorem 9 neither a nor b separates G and therefore neither $\bar{a}$ nor $\bar{b}$ separates S. But the common part of $\bar{a}$ and $\bar{b}$ consists of two mutually exclusive continua $\bar{p}$ and $\bar{q}$. It follows, by a theorem of Miss Mullikin's,‡ that $S-J$ is the sum of two mutually exclusive domains. It is easy to see that one of these domains (call it D_1)

*Proceedings of the National Academy of Sciences, vol. 10 (1924), pp. 170–175. In this paper in the statement of Lemma 5 replace the last "M" by "K". Hereafter this paper will be referred to as P. C. S.

† Loc. cit.

‡ Cf. her thesis *Certain theorems relating to plane connected point sets*, these Transactions, vol. 24 (1922), pp. 144–162.

is bounded and the other one (D_2) is unbounded. Let I denote the set of all those elements of G which are subsets of D_1 and let E denote the set of all those which are subsets of D_2. Clearly I and E are domains of elements of G and $I+E = G-J$. Let B denote the boundary of I. If B is not identical with J then it is a proper subset of J. It easily follows, with the help of Theorems 10 and 6, that B is a simple continuous arc of elements of G. But B separates G and, by Theorem 9, no arc separates G. Thus the supposition that J is not the boundary of I leads to a contradiction. In a similar way it may be proved that J is also the boundary of E.

Definition. Of the two domains which are complementary to a simple closed curve of elements of G, the bounded one will be called the *interior*, and the unbounded one will be called the *exterior*, of that curve.

Theorem 12. *If D_1 and D_2 are bounded domains of elements of G, and D_1 has a connected boundary, and the boundary of D_2 is a subset of D_1, then D_2 is a subset of D_1.*

Proof. Since D_2 is bounded it has at least one boundary element. But every one of its boundary elements belongs to D_1. Hence D_1 contains at least one element of D_2. Suppose D_2 is not a subset of D_1. Then it must contain an element which does not belong to D_1. It follows that D_2 contains an element of B_1, the boundary of D_1. But B_1 is connected and contains no point of the boundary of D_2. Hence B_1 is a subset of D_2. Thus D_2 contains the boundary of E, the unbounded complementary domain of B_1. Therefore D_2 contains an element of E. But E is connected and contains no point of the boundary of D_2. Hence E is a subset of D_2, contrary to the hypothesis that D_2 is bounded.

Theorem 13. *If R is a region (of elements of G) and k is either a single element of G or a simple continuous arc of elements of G every element of which (except possibly just one of its end elements) belongs to R, the set of all those elements of R which do not belong to k is a domain of elements of G.*

Proof. Let B denote the boundary of R. By Theorem 9, k does not separate G. Hence $\bar{k}$ does not separate S. Furthermore, if x and y are any two elements of R, $\bar{x}$ and $\bar{y}$ are not separated from each other by $\bar{B}$. Also, either $\bar{k}$ and $\bar{B}$ have no point in common or their common part is a bounded, closed and connected point set consisting of all the points of a certain single element of G. It follows* that $\bar{k}+\bar{B}$ does not separate $\bar{x}$ from $\bar{y}$. Hence $k+B$ does not separate x from y. Hence the set of all those elements of R which do not belong to k is connected. It easily follows that it is a domain.

* See S. Janiszewski, loc. cit.

THEOREM 14. *If R is a region of elements of G there exists a simple closed curve of elements of G such that every element of G which belongs to this curve is an element of R.*

Proof. Let p and q denote two distinct elements of the region R. There exists* a simple continuous arc pq of elements of G such that every element of pq belongs to R. Let r denote some element of pq distinct from p and from q. By Theorem 13, $R-r$ is a domain. Hence there exists a simple continuous arc pyq which is a subset of $R-r$ and has p and q as its end elements. It is easy to see that the sum of the arcs pxq and pyq contains as a subset a simple closed curve of elements of G.

THEOREM 15. *If pxq and pyq are simple continuous arcs (of elements of G) which have p and q as their extremities but which have in common no other element of G, and J is the simple closed curve formed by these two arcs, and pzq is a simple continuous arc (of elements of G) every element of which, except p and q, belongs to R, the interior of J, and J_1 denotes the simple closed curve formed by the arcs pxq and pzq and J_2 denotes the one formed by pyq and pzq, then* (1) *R_1, the interior of J_1, is a subset of R,* (2) *pyq is, except for p and q, wholly in the exterior of R_1,* (3) *R_1 has no point in common with R_2, the interior of J_2.*

Theorem 15 may be proved by an argument closely analogous to that used to prove Theorem 24 of F. A. In the proof there given reference is made to Theorem 21 of F. A. For the case where K and R are interiors of simple closed curves of elements of G this Theorem 21 may be easily proved with the help of Theorem 11 above.

THEOREM 16. *Under the same hypothesis as in Theorem* 15, *R is the sum of R_1, R_2, and $pzq-(p+q)$.*

Theorem 16 may be proved by an argument closely parallel to that employed in F. A. to prove Theorem 25.

THEOREM 17. *If p and q are two distinct elements of G and pxq, pyq and pzq are simple continuous arcs of elements of G no two of which have in common any element except their extremities (p and q) and J_1, J_2 and J_3 are the simple closed curves formed by these arcs taken in pairs, then the interiors of J_1, J_2 and J_3 are not mutually exclusive.*

Theorem 17 may be proved with the help of Theorems 15 and 16 by a method similar to that used in F. A. to prove Theorem 26 with the aid of Theorems 24 and 25.

THEOREM 18. *If pxq and pyq are simple continuous arcs (of elements of G) which have in common only their extremities p and q, J is the simple closed curve formed by these arcs, and pzq is an arc which lies, except for*

---

* See Theorem 15 of F. A.

its extremities, entirely in the exterior of J, then (1) *either y is without J_1, the simple closed curve formed by pxq and pzq, or x is in the exterior of J_2, the simple closed curve formed by pyq and pzq,* (2) *if y is without J_1 then x is in the interior of J_2 and the interior of J_2 is the sum of the interior of J, the interior of J_1 and the set of elements $pxq-(p+q)$.*

Theorem 18 may be proved with the help of Theorems 16 and 17 by a method analogous to that used in F. A. to prove Theorem 27 with the aid of Theorems 25 and 26.

THEOREM 19. *If R is a region (of elements of G) and p is an element of R, then there exists a simple closed curve of elements of G which lies in R and whose interior contains p and is a subset of R.*

With the use of Theorems 11, 12, 13, 17 and 18, Theorem 19 may be proved by an argument closely analogous to that employed to prove Theorem 36 in F. A.

DEFINITION. A set R of elements of G will be said to be a *region in the restricted sense* if and only if it is the interior of some simple closed curve of elements of G.

THEOREM 20. *If p is an element of G and H is a set of elements of G then p is a limit element of H if and only if every region in the restricted sense that contains p contains also an element of H distinct from p.*

Suppose first that p is a limit element of H and that R is a region in the restricted sense which contains p. Since R is also a region in the original sense, therefore, by Theorem 4, R contains an element of H distinct from p.

Suppose, secondly, that every region in the restricted sense which contains p contains an element of H distinct from p. If R is a region in the original sense that contains p, then, by Theorem 19, there exists a region in the restricted sense which contains p and which is a subset of R. Since every such region contains an element of H distinct from p, therefore so does R. Hence, by Theorem 4, p is a limit element of H.

THEOREM 21. *If the word "point", as used in F. A., is interpreted to mean "element of G" (and thus the set of all "points" is identified with the set of elements G) and the word "region", as used therein is interpreted to mean "region in the restricted sense", as defined above, then for the space S (G) consisting of all such "points" (elements of G) Axioms* 1–8 *of F. A. all hold true.*

Proof. It has been established that Axioms 1, 2, 4 and 5 and Theorem 4 of F. A. hold true for the set of elements G provided *region* is interpreted as defined near the beginning of the present section. With the help of Theorems 19 and 20 it is easy to see that these axioms continue to hold true if region is interpreted in the restricted sense as defined above. That

Axiom 3 holds true for this interpretation can easily be seen with the help of Theorem 11. The truth of Axioms 6′ and 7′ of the system Σ_2 (see page 163 of F. A.) is a consequence of Theorems 11 and 13. Thus all the axioms of the system Σ_2 hold true for G. With the aid of results established in my paper *Concerning a set of postulates for plane analysis situs** and the fact that every region in the restricted sense is the interior of a simple closed curve it easily follows that Axiom 6 and 7 also hold true.

THEOREM 22. *Between the continua of the upper semi-continuous collection G and the points of an ordinary euclidean plane S there is a one to one correspondence (with single valued inverse) which preserves limits, that is to say which has the property that a point P in S is a limit point of a point set M in S if and only if the element of G which corresponds to P is a limit element (in the sense of the definition given in this paper) of the set of elements corresponding to M.*

The truth of Theorem 22 follows from Theorems 20 and 21 and the results of my paper *Concerning a set of postulates for plane analysis situs.*

II. THE PRIME PARTS OF A BOUNDED CONTINUUM IN THE PLANE

Hans Hahn† has introduced the notion of *prime parts* of a continuum. If P is a point of a continuum M then by the prime part K_P (of M) is meant the set of all points $[X]$ belonging to M such that, for every positive number e, there exists a finite set of irregular points (of M), $X_1, X_2, X_3, \cdots, X_n$ such that

$$r(X, X_1) \leq e, \quad r(X_1, X_2) \leq e, \quad \cdots, \quad r(X_{n-1}, X_n) \leq e, \quad r(X_n, P) \leq e.‡$$

In my paper *Concerning the prime parts of a continuum*,§ I have shown that if a bounded continuum has more than one prime part then it is a

* These Transactions, vol. 20 (1919), pp. 169–178.

† *Über irreduzible Kontinua*, Sitzungsberichte der Königlichen Akademie der Wissenschaften zu Wien, vol. 130 (1921), pp. 217–250.

‡ A continuum M is said to be connected im kleinen (or regularly connected) at the point P if for every positive number e there exists a positive number d such that if X is a point of M at a distance from P less than d then X and P lie together in some connected subset of M of diameter less than e. Cf. Hans Hahn, *Über die allgemeinste ebene Punktmenge, die stetiges Bild einer Strecke ist*, Jahresbericht der Deutschen Mathematiker-Vereinigung, vol. 23 (1914), pp. 318–322; Pia Nalli, *Sopra una definizione di dominio piano limitato da una curva continua, senza punti multipli*, Rendiconti del Circolo Matematico di Palermo, vol. 32 (1911), pp. 391–401; S. Mazurkiewicz, *Sur les lignes de Jordan*, Fundamenta Mathematicae, vol. 1 (1920), pp. 166–209. An *irregular* point of a continuum M is a point at which M is not regularly connected. If X and Y are two points, $r(X, Y)$ denotes the distance from X to Y.

§ Loc. cit. Hereafter this paper will be referred to as P. C.

continuous curve with respect to its prime parts considered as points. In this section I will establish the following more general theorem.

THEOREM 23. *If, in a plane S, M is a bounded continuum no prime part of which separates S, and every prime part of M is considered as an element, and every point which does not belong to M is considered as an element, then the collection G of all such elements is an upper semi-continuous collection, and between the elements of G and the points of a plane H there exists a one to one correspondence which preserves limits and which is such that the image in H of the set of all prime parts of M is a continuous curve.*

Proof. Suppose p is an element of G which is a prime part of M and suppose that e is a positive number. By Theorem 3 of P. C. there exists a positive number d, less than e, such that if q is any element of G which is a prime part of M and whose lower distance from p is less than d then its upper distance from p is less than e. If q is any element of G which is not a prime part of M then q is a point, and therefore its upper distance from p is the same as its lower distance and thus its upper distance from p is less than e if its lower distance from p is less than d.

Suppose that p is an element of G which is not a prime part of M. In this case p is a point and if e is any positive number and d is less than e and also less than the lower distance of p from M then every element of G whose lower distance from p is less than d is also a point and therefore its upper distance from p is less than d and therefore less than e.

It follows that G is an upper semi-continuous collection. The truth of the remainder of Theorem 23 easily follows with the help of Theorem 17 of P. C. and Theorem 22 of the present paper.

Thus, regarded as being composed of its prime parts as elements, *every bounded continuum* (no one of whose prime parts separates its plane) *is a continuous curve,* not only as far as its internal structure is concerned, but *also as far as its relation to the remainder of the plane is concerned.**

* Professor R. L. Wilder has called my attention to the fact that the statement in line 10 of page 172 of P. C. S. to the effect that q contains no point of the boundary of D_x is incorrect and that Lemma 3 appears to be false. This is indeed the case if in the definition (given near the top of page 171) of the outer boundary of D *with respect to the prime parts of* M the phrase "boundary of D" is interpreted to mean the boundary, in the ordinary sense, of the *set of points* D. If, however, this phrase is interpreted to mean the boundary (as defined in Section I of the present paper) of the domain D, the points of $S-M$ (and therefore, in particular, those of D) and the prime parts of M being the elements of G, then Lemma 3 holds true and the argument given in P. C. S. to prove it holds good except that the first sentence of this argument is to be replaced by the following: "Let B denote the point set obtained by adding together all the prime parts of M which belong to K." The fourth paragraph of page 171 of P. C. S. is to be replaced by the following statement: "In view of Lemma 2 it is clear that every prime part of M which

III. The maximal connected subsets of a closed plane point set

A maximal connected subset of a point set M is a connected subset of M which is not a proper subset of any other connected subset of M.

Theorem 24. *If, in a plane S, M is a closed point set and every maximal connected subset of M is considered as an element and every point which does not belong to M is considered as an element, then the set G of all such elements is an upper semi-continuous collection.*

Proof. Let G_1 denote the set of all maximal connected subsets of M and let G_2 denote the set of all points which do not belong to M.

Suppose p is an element of G_1. Let e denote a positive number. There exists a positive number d such that if q is an element of G at a lower distance from p less than d then q is at an upper distance from p less than e. For suppose this is not the case. Then there exists a positive number e and an infinite sequence of distinct elements (of the set G) p_1, p_2, $p_3, \cdots$ such that, for every n, p_n contains two points B_n and C_n such that B_n is at a distance less than $1/n$ from some point A_n which belongs to p, while C_n is at a distance greater than e from every point of p. There exist two points B and C and a sequence of distinct integers n_1, n_2, $n_3, \cdots$ such that B is the sequential limit point of the sequence B_{n_1}, B_{n_2}, $B_{n_3}, \cdots$ and C is the sequential limit point of the sequence C_{n_1}, C_{n_2}, $C_{n_3}, \cdots$. The limiting set of the sequence p_{n_1}, p_{n_2}, $p_{n_3}, \cdots$ is a closed and connected point set K which contains B and C. But clearly B belongs to p and C does not. Since the continua p and K have B in common, their sum is connected. But their sum is a subset of M and it contains a point C not belonging to p. Thus p is not a maximal connected subset of M. But this is contrary to hypothesis.

The case where p is an element of G_2 may be treated by a method analogous to that employed in a similar connection in the proof of Theorem 23 in Section II. The truth of Theorem 23 is therefore established.

As a consequence of Theorems 24 and 22 we have the following result.

Theorem 25. *If, in a plane S, M is a closed and bounded point set no subset of which separates S, and every maximal connected subset of M is considered as an element, and every point which does not belong to M is considered as an element, then the set of all such elements is topologically equivalent to the set of all points in a plane.*

IV. Upper semi-continuous collections in space of n dimensions

Theorem 26. *If, in a euclidean space S of any number of dimensions, G is an upper semi-continuous collection of mutually exclusive continua such*

belongs to the set K defined above contains at least one point of the point set which forms the outer boundary, in the ordinary sense, of the domain (of points) D."

that every point of S belongs to some continuum of the collection G, then (regardless of whether the continua of G separate S) the term region may be so defined that (1) *if the elements of G are called points then Axioms* 1, 2, 4 *and* 5 *and Theorem* 4 *of F. A. all hold true and furthermore an element p of G will be* (*in the sense defined in* I) *a limit element of a set H of elements of G if and only if every region that contains p contains at least one element of H distinct from p.*

I will proceed to indicate how this theorem may be established.

Let us retain all the definitions given, in Section I, before the statement of Theorem 1. But let the definition of a region of elements of G, as given in that section, be replaced by the following:

Definition. If p is an element of G and e is a positive number, then by R_{pe} is meant the set of all elements q of G such that q and p belong to some connected set of elements of G such that every element of this connected set is at an upper distance from p less than e. For every p and e the set R_{pe} is called a *region* of elements of G.

It is easy to see that Theorems 1, 3 and 4 of Section I hold true here and that Axioms 1, 2, 4 and 5 and Theorem 4 of F. A. may be established as indicated after the statement of Theorem 4 in Section I.

University of Texas,
Austin, Tex.

ON THE OSCILLATION OF A CONTINUUM AT A POINT*

BY

WALLACE ALVIN WILSON

1. The concept of oscillation of a continuum at a given point was introduced by S. Mazurkiewicz in his paper on Jordan curves† and the idea has since been used frequently by other writers‡. Less attention has been given to the general concept, however, than to the special case where the oscillation is zero, a property which H. Hahn has shown to be equivalent to the property of connectedness im kleinen.

In this article several new properties of the oscillatory function are developed. In particular it is shown that the oscillation is an upper semi-continuous function and therefore has the well known characteristics of pointwise discontinuous functions. Furthermore, the behavior of the function in the neighborhood of points where its value is greater than zero is investigated. (See §§ 11–15.)

The results are obtained by the introduction of an auxiliary function $\tau(a)$ (§ 3) and oscillatory sub-sets (§§ 7–10) of a continuum, both of which have interesting properties in themselves. The work is confined to continua, although several of the theorems can be readily extended to non-closed connected sets.

2. **Notation.** The ordinary notation of the theory of aggregates is employed, with the following modifications.

If A is the common part of a system of aggregates $\{C\}$, we write $A = Dv[C]$.

If A is a real part of B, we write $A \subset B$.

If A is a part of B and may be identical with B, we write $A \subseteq B$.

3. **Definitions.** Let A be any continuum and $a \varepsilon A$. Let $V_\delta(a)$ denote the set of points of A whose distance from a is less than δ, $\delta > 0$. Let C_δ denote a subcontinuum of A which contains all points of $V_\delta(a)$. The lower bound of the diameters of all such sets C_δ, for all $\delta > 0$, is denoted by $\tau_A(a)$, or simply $\tau(a)$.

For convenience the previous sentence may be written

$$\tau(a) = \text{Min Diam } [C_\delta].$$

* Presented to the Society, September 10, 1925.

† *Sur les lignes de Jordan*, Fundamenta Mathematicae, vol. 1, pp. 166 ff.

‡ E. g., Z. Janiszewski, C. Kuratowski, and B. Knaster.

If a set is unlimited, we call its diameter ∞. If for a point a every subcontinuum C_δ is unlimited, we say that $\tau(a) = \infty$.

It is a comparatively simple matter to show the existence of the auxiliary function just defined for each point of any continuum. It is also evident that the above definition might well be employed as a definition of the oscillation of a continuum at a point. Without entering into details, it may be stated that $\tau(a)$ has all the properties of S. Mazurkiewicz' oscillatory function $\sigma(a)$ as given in the article referred to above, pp. 170–178. It is not, however, precisely the same function for continua in general, as the next section will show.

4. The function $\sigma(a)$ is defined as follows (loc. cit., p. 170). If x and y are any two points of a continuum A, let $C(x, y)$ be any subcontinuum of A containing x and y. The number $\varrho_A(x, y) = \text{Min Diam}\,[C(x, y)]$ for all possible sets $C(x, y)$ is called the relative distance between x and y with respect to A. Then $\sigma(a) = \overline{\lim}\ \varrho_A(x, y)$ as x and y approach a.

It is easy to construct continua for which $\tau(a) \neq \sigma(a)$ at certain points. However, as the following theorem shows, $\tau(a)$ and $\sigma(a)$ vanish at the same points. Hence the genre of a point is the same for both definitions of oscillation.

THEOREM. *If A is a continuum and $a \varepsilon A$, then $\sigma(a) \leqq \tau(a) \leqq 2\sigma(a)$.*

Proof. For any $\varepsilon > 0$ there exists a $\delta > 0$ and a sub-continuum C_δ of A for which

$$V_\delta(a) \subseteq C_\delta \subseteq A, \tag{1}$$

and

$$\text{Diam}\ C_\delta \leqq \tau(a) + \varepsilon. \tag{2}$$

From (2) we have at once that $\varrho_A(x, y) \leqq \tau(a) + \varepsilon$ for any pair of points x and y in $V_\delta(a)$. Therefore

$$\sigma(a) \leqq \tau(a). \tag{3}$$

On the other hand, for any $\varepsilon > 0$ there is a $\delta > 0$ such that

$$\varrho_A(x, y) \leqq \sigma(a) + \frac{\varepsilon}{2} \tag{4}$$

for any pair of points x and y in $V_\delta(a)$. By definition of $\varrho_A(x, y)$ there exists a sub-continuum $C(x, y)$ of A such that

$$\text{Diam}\ C(x, y) \leqq \varrho_A(x, y) + \frac{\varepsilon}{2} \leqq \sigma(a) + \varepsilon. \tag{5}$$

Now relations (4) and (5) hold for the particular case that $x = a$. Hence the union of all the sets $C(a, y)$ is a connected sub-set C of A which con-

tains $V_\delta(a)$ and $\bar{C}$ is a sub-continuum of A containing $V_\delta(a)$. From (5) we have

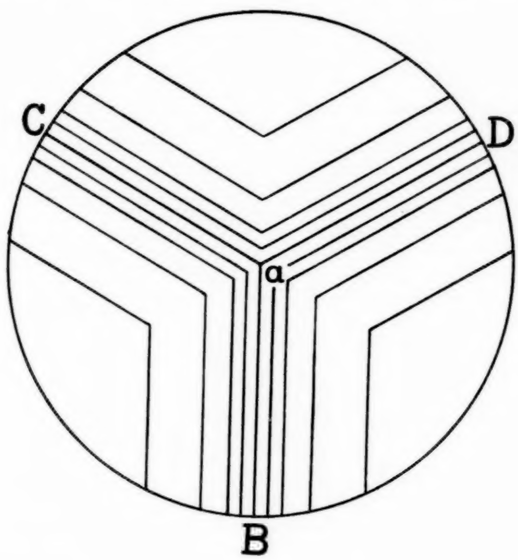
$$\text{Diam}\,\bar{C} = \text{Diam}\,C \leqq 2\sigma(a) + 2\epsilon,$$

whence

$$(6) \qquad \tau(a) \leqq 2\sigma(a) + 2\epsilon.$$

Since (6) holds for any $\epsilon > 0$, (3) and (6) give the theorem.

Example. The following is a case where $\sigma(a) \neq \tau(a)$. Draw a circle of center a and radius r and three radii trisecting the circle. On the radii bisecting each of the sectors thus formed take a sequence of points whose



distance from a is $r/2^n$, $n = 1, 2, 3, \cdots$, and through each of these points draw two lines parallel to the radii bounding the sector and terminated by the arc of the sector. Let A be the continuum consisting of the circumference, the three radii, and the series of broken lines.

Two points lying in any $V_\delta(a)$ and not lying on the radii aB, aC, or aD will lie in the same sector or in adjacent sectors, say those whose arcs are BC and CD. They can be joined by a sub-continuum passing through C and of diameter not greater than $r + 4\delta$ and not less than r. Thus $\sigma(a) = r$.

But $V_\delta(a)$ contains points in all three sectors. Hence any sub-continuum containing $V_\delta(a)$ must contain two of the points B, C, D and therefore has

a diameter at least as great as the distance $BC = r\sqrt{3}$. Thus $\tau(a) \geqq r\sqrt{3}$ and it is easy to see that $\tau(a) = r\sqrt{3}$. In this example, then, $\tau(a) = \sqrt{3}\,\sigma(a)$.

5. THEOREM. *Let $A = \{x\}$ be a continuum. Then $\tau(x)$ and $\sigma(x)$ are upper semi-continuous at all points of A.*

Proof. At points where $\tau(x)$ is infinite the theorem obviously holds. Let a be a point of A at which $\tau(a) = k \neq \infty$. Then for $\varepsilon > 0$, there is a $\delta > 0$ and a sub-continuum C_δ of A for which

(1) $$V_\delta(a) \subseteq C_\delta \subseteq A,$$

and

(2) $$\text{Diam}\, C_\delta \leqq k + \varepsilon.$$

Now for any x in $V_\delta(a)$ there is an $\eta > 0$ such that $V_\eta(x) \subseteq V_\delta(a)$. Hence

(3) $$V_\eta(x) \subseteq C_\delta.$$

Relations (1), (2), and (3) show that

$$\tau(x) \leqq k + \varepsilon = \tau(a) + \varepsilon, \quad \text{for } x \text{ in } V_\delta(a),$$

which is the definition of upper semi-continuity.

Likewise for any $\varepsilon > 0$ there is a $\delta > 0$ such that any two points x and y in $V_\delta(a)$ can be joined by a sub-continuum $C(x, y)$ of A whose diameter is not greater than $\sigma(a) + \varepsilon$. For any point x' in $V_\delta(a)$ there is an $\eta > 0$ such that $V_\eta(x') \subseteq V_\delta(a)$. Hence any two points of $V_\eta(x')$ can be joined by a sub-continuum of A of diameter not greater than $\sigma(a) + \varepsilon$.

Then we have $\sigma(x') \leqq \sigma(a) + \varepsilon$ for all x' in $V_\delta(a)$, which is the requirement for upper semi-continuity.

The above theorem shows that the oscillation of a continuum, if everywhere finite, is continuous or pointwise discontinuous, like the oscillation of a one-valued function of a real variable. Among the properties deducible from this fact may be mentioned the following.

COROLLARY 1. *Let $A = \{x\}$ be a continuum. Then $\tau(x)$ and $\sigma(x)$ are continuous at each point of A for which their value is zero.*

COROLLARY 2. *Let $A = \{x\}$ be a continuum. Then the set of points for which $\tau(x) \geqq k$, or $\sigma(x) \geqq k$, any constant, is a closed set; the set of points for which either function is discontinuous is of the first category; and the set for which either function is continuous is of the second category with respect to A.*

COROLLARY 3. *Let $A = \{x\}$ be a continuum and B be a closed limited part of A. If $\tau(x)$ or $\sigma(x)$ is finite for each point of B, it is limited in B and there is at least one point of B at which it takes on its maximum value.*

6. **Generalized irreducible continua.** We now proceed to extend the notion of a continuum irreducible between two points. Let C be a continuum and let A be any sub-set of C. The continuum C is called *irreducible about* A if there is no continuum C' satisfying the relation $A \subseteq C' \subset C$.

THEOREM. *Let C be a limited continuum and $A \subseteq C$. Then there exists at least one sub-continuum D of C which is irreducible about A.*

Proof. In order to prove this theorem it is only necessary to show, as C. Kuratowski has demonstrated,* that if C_i is a sequence of continua such that

$$C \supseteq C_1 \supseteq C_2 \supseteq \cdots \supseteq C_i \supseteq C_{i+1} \supseteq \cdots \supseteq A \tag{1}$$

and $C_\omega = Dv\,[C_i]$, then C_ω is a continuum and contains A.

Since each C_i is closed and limited, C_ω exists. By (1) $A \subseteq C_\omega$. Since all the aggregates C_i form a part of the limited aggregate C and $C_\omega \neq 0$, it follows† that C_ω is a continuum. Our theorem is therefore proved.

Without going into the properties of irreducible continua, it is perhaps well to call attention to the fact that under the conditions of the above theorem there may be many sub-continua of C irreducible about A. If C is unlimited, there may be no irreducible sub-continuum.

7. **Oscillatory sets.** Let A be a continuum and $a \in A$. Let $\delta_1 > \delta_2 > \cdots$, $\delta_i \to 0$, and let C_i denote a sub-continuum of A irreducible about $V_{\delta_i}(a)$. If the sequence $\{C_i\}$ is monotone decreasing, i. e., $C_1 \supseteq C_2 \supseteq \cdots$, we call $C = Dv\,[C_i]$ an oscillatory set of A about a.

A continuum may have many oscillatory sets about one of its points. For instance, in the example given in § 4 each of the broken lines BaC, CaD, and DaB is an oscillatory set of A about the point a. If a continuum is unlimited, it may have no oscillatory set about one or more of its points. However, we have the following theorem.

THEOREM. *Let A be a continuum, $a \in A$, and $\sigma(a)$ be finite. Then there exists at least one oscillatory set of A about a.*

Proof. Let $\delta_1 > \delta_2 > \cdots$ and $\delta_i \to 0$. Since $\sigma(a)$ is finite, so is $\tau(a)$. Hence there is a sub-continuum C_1 of A which is limited and irreducible about $V_{\delta_1}(a)$. Likewise there is a sub-continuum C_2 of C_1 which is irreducible about $V_{\delta_2}(a)$; etc. Thus we have a monotone decreasing sequence

$$C_1 \supseteq C_2 \supseteq C_3 \supseteq \cdots,$$

and $C = Dv\,[C_i]$ is an oscillatory set of A about a by definition.

* C. Kuratowski, *L'élimination des nombres transfinis*, Fundamenta Mathematicae, vol. 3, pp. 88–89.

† F. Hausdorff, *Grundzüge der Mengenlehre*, p. 302.

8. THEOREM. *Let A be a continuum and $a \varepsilon A$. If $\tau(a)$ is finite and C denotes any oscillatory set of A about a, then $\tau(a) =$* Min Diam $[C]$. *If $\tau(a) = \infty$, and any oscillatory set C of A about a exists,* Diam $C = \infty$.

Proof. Let Min Diam $[C] = k$. Then there is at least one oscillatory set C for which

$$k \leqq \text{Diam } C \leqq k + \frac{\varepsilon}{2}, \tag{1}$$

for an arbitrarily chosen positive ε. Now suppose that $C = Dv[C_i]$, where each C_i is a sub-continuum of A irreducible about $V_{\delta_i}(a)$, $\delta_i \to 0$. It follows at once that there is an i_0 such that for all $i > i_0$ every point of the set C_i has a distance from C not greater than $\varepsilon/4$. Hence

$$\text{Diam } C_i \leqq \text{Diam } C + \frac{\varepsilon}{2} \leqq k + \varepsilon. \tag{2}$$

Thus we have at least one sub-continuum C_i of A which contains a $V_\delta(a)$ and whose diameter is not greater than $k + \varepsilon$. Hence $\tau(a) \leqq k + \varepsilon$, whatever ε may be. Thus

$$\tau(a) \leqq k. \tag{3}$$

On the other hand, if $\tau(a)$ is finite, there is a limited sub-continuum B of A containing a $V_{\delta_1}(a)$ and of diameter not greater than $\tau(a) + \varepsilon$, where $\varepsilon > 0$ is arbitrary. Now B contains a sub-continuum C_1 irreducible about $V_{\delta_1}(a)$ and as in § 7 there is a monotone decreasing sequence $\{C_i\}$, whose divisor C is an oscillatory set about a. Since Diam $B \leqq \tau(a) + \varepsilon$, Diam $C \leqq \tau(a) + \varepsilon$. Thus

$$\text{Min Diam } [C] \leqq \tau(a). \tag{4}$$

Relations (3) and (4) give the theorem for the case that $\tau(a)$ is finite. The case that $\tau(a) = \infty$ is disposed of by (3).

9. **The two kinds of oscillatory sets.** We have at once as a corollary of the previous theorem that $\sigma(a) \leqq$ Min Diam $[C]$. It is also evident that these sets furnish an independent means of defining the function $\tau(a)$.

From the two previous sections it is clear that the oscillatory sets C of a continuum A may be of two kinds:

I. C is the divisor of a *descending* sequence of sub-continua $\{C_i\}$, where each C_i is irreducible about a $V_{\delta_i}(a)$, $\delta_i \to 0$.

II. C is irreducible about every $V_\delta(a)$ for every δ less than or equal to some δ_0.

In particular C is of the second kind if, in the monotone decreasing sequence $\{C_i\}$ determining C, all the sets C_i from some i_0 on are identical.

That such sets exist follows at once from the fact that if A is indecomposable there exist in any $V_\delta(a)$ points x such that A is irreducible between a and x. If, however, C is an oscillatory set of the second kind, it is not necessarily indecomposable.

If there is only a finite number of oscillatory sets of A about a, the diameter of one of them is $\tau(a)$. It is also easy to show that, if $\sigma(a) = 0$, one of the oscillatory sets of A about a is the point a itself.

10. THEOREM. *Let A be a continuum and $a \varepsilon A$. Let C be an oscillatory set about a with a finite diameter $k \neq 0$. Then for any $\varepsilon > 0$, however small, there exists a continuum of condensation of A containing a and forming a part of C whose diameter is greater than or equal to $k - \varepsilon$.*

Proof. Let $\delta_1 > \delta_2 > \cdots$ and $\delta_i \to 0$, and let $\{C_i\}$ be a monotone decreasing sequence of sub-continua of A with each C_i irreducible about $V_{\delta_i}(a)$. Let $C = Dv[C_i]$. Since C is limited, there is no loss in generality in assuming that every C_i is also limited.*

Now let x be any point of C different from a, and let h denote the distance from a to x. Let $\varepsilon < h/4$, and let Γ_σ be the interior of a sphere of center x and of radius $\sigma < \varepsilon/2$. Then, for i greater than some i_0, $\bar{\Gamma}_\sigma \cdot \bar{V}_{\delta_i}(a) = 0$. Let

$$(1) \qquad D_i = C_i - C_i \cdot \Gamma_\sigma .$$

Each D_i is closed and contains $V_{\delta_i}(a)$.

Let $D = C - C \cdot \Gamma_\sigma$ and C_σ be that component of D containing a; it also contains points of the frontier of Γ_σ, which we denote by Front Γ_σ. This follows from Janiszewski's theorem.† There is at least one point z_i of D_i in $V_{\delta_i}(a)$ not lying on C_σ, for otherwise C_i would not be irreducible about $V_{\delta_i}(a)$.

Let S_{z_i} be that component of D_i which contains one of these points z_i. As in the case of C_σ we see that S_{z_i} has a point on Front Γ_σ. Now if $S_{z_i} \cdot C_\sigma \neq 0$, then $C_\sigma \subseteq S_{z_i}$ since $C \subseteq C_i$. If this were true for every z_i

* For, since C is closed and limited, there is a finite closed sphere S such that every point of C is an inner point of S. A set C_i which is unlimited must contain points without S. Then, since it is a continuum and contains both inner and outer points of S, it must have points in common with $F = \text{Front}\, S$. Let $F_i = F \cdot C_i$. The sets F_i are closed and limited; also the sequence $\{F_i\}$ is monotone decreasing. Then either $F_i = 0$ for all i greater than some i_0 or $F_i \neq 0$ for all values of i. In the latter case $Dv[F_i] \neq 0$. But, since each $F_i \subset C_i$, $Dv[F_i] \subset C$, which makes $F \cdot C \neq 0$, contrary to our assumption regarding S.

† Janiszewski, *Sur les continus irréducibles entre deux points*, Journal de l'Ecole Polytechnique, ser. 2, vol. 16 (1912), p. 100. "Let C be a limited continuum and a be one of its points. Let A be closed and a be an inner point of A. Then there is a subcontinuum of C containing a and a frontier point of A (unless C is contained in A) and contained in A."

in $V_{\delta_i}(a)$, the union of the sets S_{z_i}, which we denote by U, would be connected and $\overline{U}$ would be a real sub-continuum of C_i containing $V_{\delta_i}(a)$. This, however, is contrary to the hypothesis that C_i is irreducible about $V_{\delta_i}(a)$. Hence in each $V_{\delta_i}(a)$ there is a point z_i for which there is a sub-continuum S_{z_i} of C_i containing z_i and a point of Front Γ_σ, but containing no point of C_σ.

As $i \to \infty$, $\delta_i \to 0$ and $z_i \to a$. Let K be the aggregate of accumulation of $\{S_{z_i}\}$. K is a continuum or reduces to the point a by virtue of a theorem of Janiszewski*. As each S_{z_i} has a point on Front Γ_σ, K also has a point x' on Front Γ_σ. Hence

$$\text{(2)} \qquad \text{Diam } K \geqq h - \sigma \geqq h - \frac{\varepsilon}{2}.$$

Since $S_{z_i} \subseteq C_i$, K is a part of every C_i. Hence $K \subseteq C$. Obviously K has no points in Γ_σ. Thus

$$\text{(3)} \qquad K \subseteq C - C \cdot \Gamma_\sigma \subseteq C_\sigma.$$

But $C_\sigma \cdot S_{z_i} = 0$; this with (3) shows that K is a continuum of condensation. We have shown then that if x is any point of C different from a there is a continuum of condensation of A which forms a part of C and which contains a and at least one point x' whose distance from x is less than $\varepsilon/2$.

Now since C is a closed set it contains two points, x and y, whose distance apart is k. If one of these is a, the previous paragraph together with relations (2) and (3) give the theorem, for then $h = k$. If not, C contains a continuum of condensation K_1 of A joining a and x', a point whose distance from x is less than $\varepsilon/2$, and a K_2 joining a and y', a point whose distance from y is less than $\varepsilon/2$. Let $K = K_1 + K_2$. Since $K_1 \cdot K_2 \neq 0$, K is a continuum of condensation. Moreover,

$$\text{Diam } K \geqq \text{Dist } (x', y') \geqq \text{Dist } (x, y) - \varepsilon \geqq k - \varepsilon.$$

11. One of the fundamental theorems in S. Mazurkiewicz' paper† is to the effect that if a point of a continuum is of the second genre, it lies on a continuum of condensation. By means of § 10 we are able to show a relation between the size of this continuum of condensation and the oscillation of the given continuum at the point.

THEOREM. *Let A be a continuum, $a \varepsilon A$, and $\sigma(a) = k$ be finite and different from zero. Then for any $\varepsilon > 0$ there exists a continuum of condensation of A containing a and of diameter greater than or equal to $k - \varepsilon$.*

* Loc. cit., p. 97.

† Loc. cit., p. 176.

Proof. This is a corollary of § 10. For $\tau(a)$ has a value between k and $2k$, and hence there is an oscillatory set of A about a whose diameter is greater than or equal to k.

12. THEOREM. *Let A be a continuum, $a \varepsilon A$, and $\sigma(a) = \infty$. Then for every $G > 0$ there exists a continuum of condensation of A containing a and of diameter greater than or equal to G.*

Proof. Since $\sigma(a) = \infty$, A is not limited. Let Γ_r be a closed sphere of center a and radius G. Let $D = A \cdot \Gamma_r$.

Let D_a be that component of D containing a. In any $V_\delta(a)$ there are points z of D not lying on D_a. For otherwise $\sigma(a) \leqq \text{Diam } D_a \leqq 2G$. Let D_z be that component of D containing one of these points. It is easily seen that D_a and each D_z have points on Front Γ_r. Obviously $D_a \cdot D_z = 0$ for every z.

Since there is a z in every $V_\delta(a)$, there is a sequence $\{z_i\}$ converging to a. Let K be the aggregate of accumulation of $\{D_{z_i}\}$. It contains a and a point on Front Γ_r. Hence

$$\text{(1)} \qquad \text{Diam } K \geqq G.$$

Since $K \cdot D_a$ contains a,

$$\text{(2)} \qquad K \subseteq D_a.$$

But $D_a \cdot D_{z_i} = 0$; hence K is a continuum of condensation. As its diameter is greater than or equal to G by (1), the theorem is proved.

13. THEOREM. *Let A be a continuum, $a \varepsilon A$, and $\sigma(a) = k$ be finite and not zero. Then for any $\varepsilon > 0$ there exists a continuum of condensation of A of diameter not less than $k/2 - \varepsilon$ containing a and contained in an oscillatory set C of A about a, and having no point where the oscillation is zero.*

Proof. Let $\sigma = k/2 - \varepsilon/2$ and $\varrho = k/2 - \varepsilon$. Let Γ_σ and Γ_ϱ be closed spheres of center a and radii σ and ϱ respectively. Let A_a be that component of $A \cdot \Gamma_\sigma$ which contains a. For any $\delta > 0$ there are points z of $V_\delta(a)$ which do not lie on A_a. For otherwise $\sigma(a) \leqq \text{Diam } A_a \leqq k - \varepsilon$. Let A_z be that component of $A \cdot \Gamma_\sigma$ which contains z. It is obvious that A_a and each A_z contains one or more points on Front Γ_σ and that

$$\text{(1)} \qquad A_a \cdot A_z = 0.$$

Since $\sigma(a)$ is finite, there is a monotone decreasing sequence of limited sub-continua $\{C_i\}$ of A, each C_i irreducible about a $V_{\delta_i}(a)$, $\delta_i \to 0$, and $C = Dv\,[C_i]$ is an oscillatory set of A about a of diameter not less than k. Let $C_{1,a}$ be that component of $C_1 \cdot \Gamma_\sigma$ containing a, and $C_{1,z}$ be that com-

ponent containing z, where z is one of the points of the previous paragraph. Since Diam $C_1 \geq k$, each of these components has points on Front Γ_σ. Also, since $C_1 \subseteq A$, we have

$$(2) \qquad C_{1,a} \subseteq A_a \quad \text{and} \quad C_{1,z} \subseteq A_z.$$

Now let $\{z_i\}$ be a sequence of the points z converging to a. Let K' be the aggregate of accumulation of the sets $\{C_{1,z_i}\}$. Then K' is a continuum containing a and a point on Front Γ_σ. Since $C_1 \cdot \Gamma_\sigma$ is closed, $K' \subseteq C_1 \cdot \Gamma_\sigma$. Since $K' \cdot C_{1,a} \supseteq a$, $K' \subseteq C_{1,a}$. But $C_{1,a} \cdot C_{1,z_i} = 0$. Hence K' is a continuum of condensation.

Now let K_1 be the saturated sub-continuum of K' contained in Γ_ϱ and containing a. It has a point on Front Γ_ϱ, but no points of $\Gamma_\sigma - \Gamma_\varrho$. Hence Diam $K_1 \geq \varrho = k/2 - \varepsilon$. If $\sigma(x) = 0$ for some point of K_1, there is an $\eta > 0$ such that $V_\eta(x)$ lies in a sub-continuum $C(x)$ of A of diameter not greater than $\varepsilon/4$. Since $\sigma - \varrho = \varepsilon/2$, $C(x) \subseteq A \cdot \Gamma_\sigma$ and hence

$$(3) \qquad V_\eta(x) \subset C(x) \subseteq A_a.$$

But each point of K_1 is a point of accumulation of the sets $\{C_{1,z_i}\}$ and hence every $V_\eta(x)$ contains a point of some C_{1,z_i}. Thus we have $A_a \cdot C_{1,z_i} \neq 0$, and $C_{1,z_i} \subseteq A_a$. This, however, contradicts relations (1) and (2). Hence $\sigma(x) \neq 0$ for every point in K_1.

Since $\delta_2 < \delta_1$ and $C_2 \subseteq C_1$, the same argument applied to C_2 will yield a continuum of condensation K_2, which is a part of K_1 and of $C_{2,a}$, contains a and a point of Front Γ_ϱ, and has no point for which $\sigma(x) = 0$.

Let this be continued and set K equal to the divisor of the monotone decreasing sequence $\{K_i\}$. Since every $K_i \subseteq C_i$, $K \subseteq C$. Since $\sigma(x) \neq 0$ in every K_i, $\sigma(x) \neq 0$ in K. Since every continuum K_i contains a and a point on Front Γ_ϱ, and is limited, the divisor K is a continuum of diameter not less than $k/2 - \varepsilon$. Since every point of K_1 is a limit point of points not in K_1 and $K \subseteq K_1$, this is also true of K, and therefore K is a continuum of condensation.

It should be noted that the proof can be considerably shortened, if we do not attempt to prove that the continuum of condensation is a part of C.

14. Theorem. *Let A be a continuum, $a \in A$, and $\sigma(a) = \infty$. Then for any $G > 0$ there is a continuum of condensation of A containing a and of diameter not less than G, at each point of which $\sigma(x) > 0$.*

Proof. The theorem may be established by reasoning similar to that in § 13, by taking $\sigma = G$ and substituting for C_1 the set A itself.

15. THEOREM. *Let A be a continuum, $a \varepsilon A$, and $\tau(a) = k$, finite and not zero. Then a lies on a continuum of condensation K of A such that as $x \to a$ on K, $\overline{\lim}\, \tau(x) \leqq k$ and $\underline{\lim}\, \tau(x) \geqq k/2$.*

Proof. The first statement holds by virtue of § 5. To prove the second let K be the continuum of condensation obtained in § 13. Now if $\underline{\lim}\, \tau(x) < k/2$ as $x \to a$ on K, there is in every $V_\delta(a)$ a point x on K, such that for any ε sufficiently small and positive

$$\text{(1)} \qquad \tau(x) = \lambda < \frac{k}{2} - \varepsilon.$$

Now there is an $\eta > 0$ so small that $V_\eta(x)$ lies in a sub-continuum $C(x)$ of A of diameter not greater than $\lambda + \varepsilon/4$. If $z \varepsilon C(x)$,

$$\text{(2)} \qquad \begin{aligned} \text{Dist}\ (a, z) &\leqq \text{Dist}\ (a, x) + \text{Dist}\ (x, z) \\ &\leqq \delta + \lambda + \frac{\varepsilon}{4} \\ &\leqq \lambda + \frac{\varepsilon}{2} \\ &\leqq \frac{k}{2} - \frac{\varepsilon}{2}, \end{aligned}$$

if $\delta \leqq \varepsilon/4$. This shows that $C(x)$ lies in the closed sphere Γ_σ of § 13, since $\sigma = k/2 - \varepsilon/2$. Hence $C(x) \subseteq A \cdot \Gamma_\sigma$, and therefore

$$\text{(3)} \qquad C(x) \subseteq A_a.$$

But, since $x \varepsilon K$, every $V_\eta(x)$ contains a point of some C_{1,z_i}, which would give $A_a \cdot C_{1,z_i} \neq 0$, and as in § 13 we arrive immediately at the contradiction that $C_{1,z_i} \subseteq A_a$.

Hence the assumption that $\underline{\lim}\, \tau(a)$ can be less than $k/2$ is false.

The above theorem can at once be expressed in terms of $\sigma(x)$ by means of § 4. That $\underline{\lim}\, \tau(x)$ can equal $\frac{1}{2}\tau(a)$ is seen by considering the continuum defined as follows. For $x = 0$, y has all the values between ± 1; for $x \neq 0$, $y = \pm \sin^2(1/x)$ according as x is positive or negative. If we set $a = (0, 0)$, the segment of the y-axis between ± 1 is a continuum of condensation containing a and $\tau(a) = 2$, but as $z \to a$ on this continuum of condensation, $\underline{\lim}\, \tau(z) = 1$.

16. The example just given brings out clearly the analogy between the oscillation of a continuum and the oscillation of certain functions of a real variable. If in the definition of the continuum we set $y = 0$ when $x = 0$,

then $y = f(x)$ is a one-valued function, continuous except at $x = 0$. At points of continuity the oscillation of the function is 0; at the origin it is 2. At points of the continuum where $x \neq 0$, $\tau(x) = 0$; at the point (0,0) $\tau(x) = 2$. The oscillatory set at the point (0,0) corresponds to the aggregate of limiting values of $f(x)$ as $x \to 0$.

If we consider the oscillatory set as a tool in studying continua (as the oscillation of a function is used in studying discontinuous functions), the question at once arises as to whether it possesses any merits differentiating it from continua of condensation. Two are evident from the contents of this paper. First, the diameter of an oscillatory set is at least as great as the oscillation $\sigma(x)$ or $\tau(x)$; this is not in general true for continua of condensation, as can be shown by an example. Second, for such a continuum as the unit square the oscillation at every point is zero and the oscillatory set is the point itself, while any linear continuum containing the point is a continuum of condensation. The obvious defect is that, like the continuum of condensation, the oscillatory set at a point is not unique. This is to be expected on account of the generality of the continua considered here. With the proper restrictions this difficulty should disappear. For example, when a continuum is irreducible between two points and limited, the oscillatory set is unique, as will be shown in a paper at present under preparation.

YALE UNIVERSITY,
NEW HAVEN, CONN.

ON THE CONDITIONS OF INTEGRABILITY OF COVARIANT DIFFERENTIAL EQUATIONS*

BY

J. A. SCHOUTEN

In differential geometry conditions of integrability frequently occur, but in the cases usually investigated only the first of these conditions has to be considered. In 1922† Eisenhart and Veblen gave a necessary and sufficient condition that a geometry of paths be a Riemann geometry by using a new method of treating the conditions of integrability of higher order. Recently Veblen and Thomas have generalised this method in these Transactions‡ and succeeded in giving a very elegant treatment of linear equations of the form

$$\nabla_\mu v_{\lambda_1 \cdots \lambda_p} = 0$$

and of linear equations of the form

$$\nabla_{(\mu} v_{\lambda_1 \cdots \lambda_p)} = 0$$

for $p = 2$.§

Now the conditions of integrability have been the subject of a great number of investigations, especially by Lie, Bouquet, Mayer, Bourlet, Delassus and Riquier, and in these papers the problem is solved for very general forms of systems of differential equations. The methods used by these authors, however, are not directly applicable in the case of *covariant* equations as used in differential geometry, as they are too general and not in covariant form. Therefore it must be possible to establish a method in covariant form for the treatment of covariant equations much more simple than these general methods, and more convenient for use. In the first part of this paper (pp. 442–453) we deal with equations of the form

$$\nabla_\mu v_{\lambda_1 \cdots \lambda_p} = w_{\mu\lambda_1 \cdots \lambda_p},$$

the right side being a function of x^ν and $v_{\lambda_1 \cdots \lambda_p}$ only, containing no derivatives of $v_{\lambda_1 \cdots \lambda_p}$. The general solution of these equations, if existing,

* Presented to the Society, September 11, 1925.

† Proceedings of the National Academy of Sciences, vol. 8 (1922), pp. 19–23.

‡ Vol. 25, pp. 551–608.

§ We use in this paper the notations of the author's *Der Ricci Kalkül*, Berlin, J. Springer, 1924; cited in the sequel as R. K.

depends always on a finite number of arbitrary constants. The conditions of integrability and their treatment are given in covariant form. The linear case, including as a special case the first equation of Veblen and Thomas, is treated more in detail in a separate section. In the second part (pp. 453–473) we deal with equations of the form

$$P \nabla_\mu v_{\lambda_1 \cdots \lambda_p} = w_{\mu\lambda_1 \cdots \lambda_p},$$

the right side having the properties mentioned above and P being any operator linear homogeneous in the permutations of the $p+1$ suffixes $\mu \lambda_1 \cdots \lambda_p$. The treatment of such equations is possible by using the author's development of arbitrary quantities of degree $p+1$ in series of indivisible quantities. By using this development and a general theorem of Lie we find the necessary and sufficient condition that the general solution depend on a finite number of arbitrary constants. This condition being satisfied, the conditions of integrability and their treatment can be given in covariant form. The linear case, including as a special case the second equation of Veblen and Thomas, is treated more in detail in a separate section.

The case in which the general solution involves arbitrary functions is not dealt with in this paper.

A. On covariant equations admitting a solution for the first derivatives

1. The conditions of integrability. In this chapter we consider covariant equations of the form

$$\nabla_\mu v_{\lambda_1 \cdots \lambda_p} = w_{\mu\lambda_1 \cdots \lambda_p}; \qquad \mu, \lambda_1, \cdots, \lambda_p = a_1, \cdots, a_n, \tag{1.1}$$

$w_{\mu\lambda_1 \cdots \lambda_p}$ being a quantity, depending on x^ν and $v_{\lambda_1 \cdots \lambda_p}$ only, not containing derivatives of $v_{\lambda_1 \cdots \lambda_p}$; ∇_μ is the operator of covariant differentiation, defined by the equation*

$$\nabla_\mu v_{\lambda_1 \cdots \lambda_p} = \frac{\partial v_{\lambda_1 \cdots \lambda_p}}{\partial x^\mu} + \sum_x^{1,\cdots,p} \Gamma^\nu_{\lambda_x \mu} v_{\lambda_1 \cdots \lambda_{x-1} \nu \lambda_{x+1} \cdots \lambda_p}, \tag{1.2}$$

the parameters $\Gamma^\nu_{\lambda\mu}$ being once for all given as functions of x^ν. In (1.1) only lower suffixes appear, but all properties derived in this chapter hold also for quantities with higher and lower suffixes.

* Signs of summation are *always* omitted when they belong to *Greek* suffixes, but *never* when they belong to *Latin* ones.

Temporarily we make use of suffixes u, v, w, taking all values from 1 to $N=n^p$, and write

$$(1.3)\qquad \begin{aligned} v_{a_1\ldots a_1} &= v_1, \\ v_{a_1\ldots a_1 a_2} &= v_2, \text{ etc.,} \end{aligned}$$

so that (1.1) is equivalent to

$$(1.4)\qquad \frac{\partial v_u}{\partial x^\mu} = W_{\mu u} \quad (u = 1, \cdots, N;\ \mu = a_1, \cdots, a_n).$$

We recapitulate briefly some well known properties of these equations.* The solutions of (1.4) satisfy the equations

$$(1.5)\qquad \frac{\partial W_{\mu u}}{\partial x^\omega} + \sum_v \frac{\partial W_{\mu u}}{\partial v_v} W_{\omega v} = \frac{\partial W_{\omega u}}{\partial x^\mu} + \sum_v \frac{\partial W_{\omega u}}{\partial v_v} W_{\mu v},$$

which do not contain any derivatives of v_u and may therefore be put in the form

$$(1.6)\qquad F_x(v_1, \cdots, v_N) = 0 \qquad \left(x = 1, \cdots, N\binom{n}{2}\right).$$

If (1.6) is identically satisfied, viz. for all the values of v_u, then (1.4) is *completely integrable.* In this case if $W_{\mu u}$ are functions of x^ν, v_u, regular in the vicinity of some arbitrary given values $\overset{0}{x^\nu}$, $\overset{0}{v_u}$, there exists one and only one system of solutions v_u, regular in this region and taking the values $\overset{0}{v_u}$ for $x^\nu = \overset{0}{x^\nu}$. Hence if (1.4) are completely integrable, the general solution depends on a finite number of arbitrary constants.

If equations (1.6) are *not* identically satisfied, there will be some of them, say $N-N'$, independent. Then, by (1.6), $N-N'$ of the variables v_u are functions of the other N' and of x^ν. Without loss of generality we may assume that these variables are $v_{N'+1}, \cdots, v_N$. Then (1.6) is equivalent to

$$(1.7)\qquad v_{N'+s} = \varphi_s(v_1, \cdots, v_{N'}) \quad (s = 1, \cdots, N-N').$$

Substituting the values of $v_{N'+s}$ from (1.7) into (1.4) we get from the first N' equations N' new equations

---

* Bouquet, Bulletin des Sciences Mathématiques et Astronomiques, vol. 3 (1872), pp. 265–274; Mayer, the same Bulletin, vol. 11 (1876); Mathematische Annalen, vol. 5 (1872), pp. 448–470; Bourlet, Annales de l'École Normale Supérieure, ser. 3, vol. 8 (1891), supplément, pp. 1–63.

$$\frac{\partial v_a}{\partial x^\mu} = W'_{\mu a} \qquad (a = 1, \cdots, N'), \tag{1.4'}$$

in which $W'_{\mu a}$ contains only $v_1, \cdots, v_{N'}$. The other $N-N'$ equations (1.4) lead to $N-N'$ equations containing only $v_1, \cdots, v_{N'}$ and no derivatives:

$$F'_s(v_1, \cdots, v_{N'}) = 0 \qquad (s = 1, \cdots, N-N'). \tag{1.6'}$$

The conditions of integrability of (1.4′) are obtained by substituting the values from (1.7) into (1.5). Hence the equations (1.4′) are then and only then completely integrable when (1.6′) is satisfied *identically*.

If equations (1.6′) are not identically satisfied, there will be some of them, say $N'-N''$, independent. Then, by (1.6′), $N'-N''$ of the variables can be eliminated, giving rise to the equations

$$\frac{\partial v_b}{\partial x^\mu} = W''_{\mu b} \qquad (b = 1, \cdots, N''), \tag{1.4''}$$

$$F''_t(v_1, \cdots, v_{N''}) = 0 \quad (t = 1, \cdots, N'-N''), \tag{1.6''}$$

$W''_{\mu b}$ containing only $v_1, \cdots, v_{N''}$. The equations (1.4″, 6″) can be treated in the same way as (1.4′, 6′). Proceeding in this way we may arrive at a completely integrable system of N^* equations,

$$\frac{\partial v_c}{\partial x^\mu} = W^*_{\mu c} \qquad (c = 1, \cdots, N^*), \tag{1.4*}$$

$W^*_{\mu c}$ containing only $v_1, \cdots, v_{N^*}$, and $N-N^*$ equations, expressing the other $N-N^*$ variables as functions of $v_1, \cdots, v_{N^*}$:

$$v_d = f_d(v_1, \cdots, v_{N^*}) \quad (d = N^*+1, \cdots, N). \tag{1.8}$$

In this case the general solution of (1.4) depends on a finite number of arbitrary constants. But it is also possible that we may arrive at a number of independent equations between $v_1, \cdots, v_N$ greater than N. Then (1.4) is inconsistent.

2. **The covariant form of the conditions.** We will now derive a covariant form of the conditions of integrability, much more convenient than the form deduced in the preceding section. To this purpose we consider once more the process leading from (1.4,6) to (1.4′, 6′). (1.4′) and (1.6′) are algebraic consequences of (1.6) and the first derivative of (1.6),

$$\frac{\partial F_x}{\partial x^\omega} = 0, \tag{2.1}$$

if in these latter equations $\partial v_u / \partial x^\mu$ is replaced by $W_{\mu u}$. (1.6′) is then and only then identically satisfied, when (2.1) is an *algebraic* consequence of (1.6). In the same way (1.4″) and (1.6″) are algebraic consequences of (1.6), (2.1) and

$$\text{(2.2)} \qquad \frac{\partial}{\partial x^\xi} \frac{\partial F_x}{\partial x^\omega} = 0,$$

if in this latter equation all derivatives $\partial v_u / \partial x^\mu$ are replaced by $W_{\mu u}$ and (1.6″) is identically satisfied when (2.2) is an algebraic consequence of (1.6) and (2.1). The process comes to an end when one of the equations derived from (1.6) by differentiation is an algebraic consequence of all preceding ones, or, when the number of independent equations between the v_u becomes greater than N.

Considering the solution of (1.4) in this way we are now able to put the condition of integrability of the covariant equation (1.1) equivalent to (1.4) into a covariant form. (1.6) is equivalent to

$$\text{(2.3)} \qquad \tfrac{1}{2} \sum_{x}^{1,\dots,p} R_{\mu_2\mu_1\lambda_x}^{\cdot\;\cdot\;\cdot\;\nu} v_{\lambda_1 \dots \lambda_{x-1} \nu \lambda_{x+1} \dots \lambda_p} = \nabla_{[\mu_2} w_{\mu_1]\lambda_1 \dots \lambda_p}$$

when on the right side always first derivatives of $v_{\lambda_1 \dots \lambda_p}$ are eliminated by means of (1.1). If we make use of the quantity

$$\text{(2.4)} \qquad R_{\mu_2\mu_1\lambda_1 \dots \lambda_p}^{\cdot\;\cdot\;\cdot\;\;\cdot\;\nu_1 \dots \nu_p} = \sum_{x}^{1,\dots,p} R_{\mu_2\mu_1\lambda_x}^{\cdot\;\cdot\;\cdot\;\nu_x} A_{\lambda_1 \dots \lambda_{x-1} \lambda_{x+1} \dots \lambda_p}^{\nu_1 \dots \nu_{x-1} \nu_{x+1} \dots \nu_p},$$

(2.3) can be written

$$\text{(2.5)} \qquad \tfrac{1}{2} R_{\mu_2\mu_1\lambda_1 \dots \lambda_p}^{\cdot\;\cdot\;\cdot\;\;\cdot\;\nu_1 \dots \nu_p} v_{\nu_1 \dots \nu_p} = \nabla_{[\mu_2} w_{\mu_1]\lambda_1 \dots \lambda_p}.$$

The present investigation requiring a great many of suffixes we will use an abridged notation, writing λ_p instead of $\lambda_p \cdots \lambda_{q+1}$, λ_q being the next suffix of the λ-series occurring *explicitly* in the same term. So v_{λ_p} stands for $v_{\lambda_p \dots \lambda_1}$, $v_{\lambda_p} w_{\lambda_q}$ $(p > q)$ for $v_{\lambda_p \dots \lambda_{q+1}} w_{\lambda_q \dots \lambda_1}$, $v_{\lambda_p} w_{\lambda_q}$ $(p < q)$ for $v_{\lambda_p \dots \lambda_1} w_{\lambda_q \dots \lambda_{p+1}}$, and $\nabla_{\lambda_p} w_{\lambda_q}$ $(p > q)$ for $\nabla_{\lambda_p \dots \lambda_{q+1}} w_{\lambda_q \dots \lambda_1} = \nabla_{\lambda_p} \nabla_{\lambda_{p-1}} \cdots \nabla_{\lambda_{q+1}} w_{\lambda_q \dots \lambda_1}$. This convention will be followed throughout this paper. Then (2.5) becomes

$$\text{(2.6)} \qquad \tfrac{1}{2} R_{\mu_2\lambda_p}^{\cdot\;\cdot\;\nu_p} v_{\nu_p} = \nabla_{[\mu_2} w_{\mu_1]\lambda_p}.$$

By covariant differentiation of (2.6), always eliminating first derivatives of $v_{\mu\lambda_p}$, we get an infinite series of equations of the form

$$\overset{y}{S}_{\mu_y\lambda_p} + \overset{y}{S}{}_{\mu_y\lambda_p}^{\cdot\,\cdot\,\nu_p} v_{\nu_p} = 0 \qquad (y = 2, 3, \cdots), \tag{2.7}$$

where the quantities S are functions of x^ν and v_{ν_p}, not containing derivatives of v_{ν_p}. The first of these equations is (2.6) written in another way. Hence this equation is equivalent to (1.6). The first and second equations (2.7) are equivalent to (1.6) and (2.1), the first three equations (2.7) are equivalent to (1.6), (2.1) and (2.2), etc. The equations (2.7) containing only v_{λ_p} and no derivatives, either are inconsistent, or there exists a number q, so that (2.7_{q+1}) is an algebraic consequence of the preceding equations. Then the same holds for (2.7_{q+2}) etc.

We have therefore proved the following theorem:

The first condition of integrability of the equation

$$\nabla_\mu v_{\lambda_p} = w_{\mu\lambda_p} \tag{A}$$

where $w_{\mu\lambda_p}$ contains only x^ν and v_{λ_p} and no derivatives, is found by covariant differentiation and alternation of (A):

$$\tfrac{1}{2} R_{\mu_2\lambda_p}^{\cdot\,\cdot\,\nu_p} v_{\nu_p} = \nabla_{[\mu_2} w_{\mu_1]\lambda_p}. \tag{B_1}$$

The other conditions are found by covariant differentiation of (B_1), eliminating every time all derivatives of v_{λ_p}:

$$\overset{y}{S}_{\mu_y\lambda_p} + \overset{y}{S}{}_{\mu_y\lambda_p}^{\cdot\,\cdot\,\nu_p} v_{\nu_p} = 0 \qquad (y = 3, 4, \cdots). \tag{B_{y-1}}$$

The quantities S contain only x^ν and v_{λ_p} and no derivatives of v_{λ_p}. Either the system (B) is inconsistent, or there exists a number q, such that (B_{q+1}) is an algebraic consequence of $(B_1, \cdots, B_q)$. In the first case (A) admits no solution; in the second case (B_{q+2}) etc. also are algebraic consequences of $(B_1, \cdots, B_q)$ and the general solution of (A) depends on a finite number of arbitrary constants.

3. The linear case. When $w_{\mu\lambda_p}$ contains v_{λ_p} only linearly, (1.1) takes the form*

$$\nabla_\mu v_{\lambda_p} = \overset{0}{u}_{\mu\lambda_p} + u_{\mu\lambda_p}^{\cdot\,\cdot\,\nu_p} v_{\nu_p}, \tag{3.1}$$

$\overset{0}{u}_{\mu\lambda_p}$ and $u_{\mu\lambda_p}^{\cdot\,\cdot\,\nu_p}$ containing only x^ν and neither v_{λ_p} nor derivatives of v_{λ_p}. The conditions of integrability are

* The case in which the right side of (3.1) reduces to zero is treated by Veblen and Thomas, these Transactions, vol. 25 (1923), pp. 584 ff.

$$\overset{0}{T}_{\mu_y\lambda_p} + T_{\mu_y\lambda_p}^{\cdot\;\cdot\;\nu_p} v_{\nu_p} = 0 \qquad (y = 2, 3, \cdots), \tag{3.2}$$

where

$$\begin{aligned} \overset{0}{T}_{\mu_2\lambda_p} &= -\nabla_{[\mu_2} \overset{0}{u}_{\mu_1]\lambda_p} - u_{[\mu_1|\lambda_p|}^{\cdot\;\cdot\;\nu_p} \overset{0}{u}_{\mu_2]\nu_p}, \\ \overset{0}{T}_{\mu_{y+1}\lambda_p} &= \nabla_{\mu_{y+1}} \overset{0}{T}_{\mu_y\lambda_p} + T_{\mu_y\lambda_p}^{\cdot\;\cdot\;\nu_p} \overset{0}{u}_{\mu_{y+1}\nu_p}, \\ T_{\mu_2\lambda_p}^{\cdot\;\cdot\;\nu_p} &= \tfrac{1}{2} R_{\mu_2\lambda_p}^{\cdot\;\cdot\;\nu_p} - \nabla_{[\mu_2} u_{\mu_1]\lambda_p}^{\cdot\;\cdot\;\nu_p} + u_{[\mu_2|\lambda_p|}^{\cdot\;\cdot\;\alpha_p} u_{\mu_1]\alpha_p}^{\cdot\;\cdot\;\nu_p}, \\ T_{\mu_{y+1}\lambda_p}^{\cdot\;\cdot\;\nu_p} &= \nabla_{\mu_{y+1}} T_{\mu_y\lambda_p}^{\cdot\;\cdot\;\nu_p} + T_{\mu_y\lambda_p}^{\cdot\;\cdot\;\alpha_p} u_{\mu_{y+1}\alpha_p}^{\cdot\;\cdot\;\nu_p} \qquad (y = 2, 3, \cdots). \end{aligned} \tag{3.3}$$

Now we assume that the first q equations (3.2) admit a solution $\overset{0}{v}_{\lambda_p}$. Then every solution of these equations has the form

$$v_{\lambda_p} = \overset{0}{v}_{\lambda_p} + \underset{1}{\alpha} \overset{1}{v}_{\lambda_p} + \cdots + \underset{s}{\alpha} \overset{s}{v}_{\lambda_p}, \tag{3.4}$$

where $\underset{1}{\alpha}, \cdots, \underset{s}{\alpha}$ are arbitrary parameters, functions of x^ν, and where $\overset{1}{v}_{\lambda_p}, \cdots, \overset{s}{v}_{\lambda_p}$ form a *fundamental set* of solutions of the *reduced* system (3.2), viz. the system deduced from (3.2) by setting $\overset{0}{u}_{\mu\lambda_p} = 0$. In the reduced system all quantities T with a suffix 0 vanish. Further we assume that the $(q+1)$th equation (3.2) is an an algebraic consequence of the preceding ones. When the system (3.2) is algebraically consistent there certainly will be a number q with these properties.

Replacing v_{λ_p} by $\overset{0}{v}_{\lambda_p}$ in the first q equations (3.2) we have by differentiation

$$\nabla_\omega \overset{0}{T}_{\mu_y\lambda_p} + \left(\nabla_\omega T_{\mu_y\lambda_p}^{\cdot\;\cdot\;\nu_p}\right) \overset{0}{v}_\nu + T_{\mu_y\lambda_p}^{\cdot\;\cdot\;\nu_p} \nabla_\omega \overset{0}{v}_\nu = 0 \quad (y = 2, \cdots, q+1). \tag{3.5}$$

Subtracting these equations from the second to the $(q+1)$th equations (3.2) (inclusive), we have, by (3.3),

$$T_{\mu_y\lambda_p}^{\cdot\;\cdot\;\nu_p} \left(\overset{0}{u}_{\omega\nu_p} + u_{\omega\nu_p}^{\cdot\;\cdot\;\alpha_p} \overset{0}{v}_{\alpha_p} - \nabla_\omega \overset{0}{v}_{\nu_p}\right) = 0 \quad (y = 2, \cdots, q+1). \tag{3.6}$$

Hence

$$a^\omega \left(\overset{0}{u}_{\omega\nu_p} + u_{\omega\nu_q}^{\cdot\;\cdot\;\alpha_p} \overset{0}{v}_{\alpha_p} - \nabla_\omega \overset{0}{v}_{\nu_p}\right),$$

where a^ω is any arbitrary vector, is a solution of the *reduced* equation (3.2). Therefore, $\overset{1}{v}_{\lambda_p}, \cdots, \overset{s}{v}_{\lambda_p}$ being a fundamental set of solutions of the reduced equations (3.2), there must exist an equation of the form

$$\overset{0}{u}_{\mu\lambda_p} + u_{\mu\lambda_p}^{\cdot\;\cdot\;\nu_p} \overset{0}{v}_{\nu_p} - \nabla_\mu \overset{0}{v}_{\lambda_p} = \sum_b^{1,\cdots,s} \overset{0b}{p}_\mu \overset{b}{v}_{\lambda_p}. \tag{3.7}$$

In the same way we prove, putting $\overset{a}{v}_{\lambda_p}$, $a = 1, \cdots, s$, in the *reduced* equations (3.2), the existence of equations of the form

$$u_{\mu\lambda_p}^{\cdot\cdot\,\nu_p} \overset{a}{v}_{\nu_p} - \nabla_\mu \overset{a}{v}_{\lambda_p} = \sum_b^{1,\cdots,s} \overset{ab}{p}_\mu \overset{b}{v}_{\lambda_p}. \tag{3.8}$$

By differentiation and alternation of (3.7, 8) we obtain

$$\begin{aligned} &\overset{i}{\varepsilon} \nabla_{[\omega} \overset{0}{u}_{\mu]\lambda_p} + \nabla_{[\omega} u_{\mu]\lambda_p}^{\cdot\cdot\,\nu_p} \overset{i}{v}_{\nu_p} + u_{[\mu|\lambda_p|}^{\cdot\cdot\,\nu_p} \nabla_{\omega]} \overset{i}{v}_{\nu_p} - \tfrac{1}{2} R_{\omega\mu\lambda_p}^{\cdot\cdot\cdot\,\nu_p} \overset{i}{v}_{\lambda_p} \\ &= \sum_b^{1,\cdots,s} (\nabla_{[\omega} \overset{ib}{p}_{\mu]}) \overset{b}{v}_{\lambda_p} + \sum_b^{1,\cdots,s} \overset{ib}{p}_{[\mu} \nabla_{\omega]} \overset{b}{v}_{\lambda_p} \quad \left(\overset{i}{\varepsilon} = \begin{cases} 1, & i = 0 \\ 0, & i \neq 0 \end{cases}; \; i = 0, 1, \cdots, s \right), \end{aligned} \tag{3.9}$$

and these equations pass by means of (3.2, 3, 7, 8) into

$$\nabla_{[\omega} \overset{ia}{p}_{\mu]} + \sum_b \overset{ib}{p}_{[\omega} \overset{ba}{p}_{\mu]} = 0 \qquad (i = 0, 1, \cdots, s). \tag{3.10}$$

Now we will prove that the general solution of (3.1) has the form (3.4), and that the parameters $\underset{1}{\alpha}, \cdots, \underset{s}{\alpha}$ can be found by the integration of a *completely integrable* system. In order that (3.1) be satisfied by (3.4) it is necessary and sufficient that

$$\begin{aligned} \nabla_\mu \overset{0}{u}_{\lambda_p} + \sum_a^{1,\cdots,s} (\nabla_\mu \underset{a}{\alpha}) \overset{a}{v}_{\lambda_p} + \sum_a^{1,\cdots,s} \underset{a}{\alpha} \nabla_\mu \overset{a}{v}_{\lambda_p} = \overset{0}{u}_{\mu\lambda_p} + \sum_a^{1,\cdots,s} \underset{a}{\alpha}\, u_{\mu\lambda_p}^{\cdot\cdot\,\nu_p} \overset{a}{v}_{\nu_p} \\ + u_{\mu\lambda_p}^{\cdot\cdot\,\nu_p} \overset{0}{v}_{\nu_p}, \end{aligned} \tag{3.11}$$

or, because of (3.7, 8),

$$- \sum_b^{1,\cdots,s} \overset{0b}{p}_\mu \overset{b}{v}_{\lambda_p} + \sum_a^{1,\cdots,s} (\nabla_\mu \underset{a}{\alpha}) \overset{a}{v}_{\lambda_p} - \sum_{a,b}^{1,\cdots,s} \underset{a}{\alpha} \overset{ab}{p}_\mu \overset{b}{v}_{\lambda_p} = 0, \tag{3.12}$$

or

$$\nabla_\lambda \underset{a}{\alpha} = \overset{0a}{p}_\lambda + \sum_b^{1,\cdots,s} \underset{b}{\alpha} \overset{ba}{p}_\lambda. \tag{3.13}$$

The conditions of integrability of (3.13) are

$$\nabla_{[\mu} \overset{0a}{p}_{\lambda]} + \sum_b^{1,\cdots,s} \overset{0b}{p}_{[\mu} \overset{ba}{p}_{\lambda]} + \sum_b^{1,\cdots,s} \underset{b}{\alpha} \left\{ \nabla_{[\mu} \overset{ba}{p}_{\lambda]} + \sum_c^{1,\cdots,s} \overset{bc}{p}_{[\mu} \overset{ca}{p}_{\lambda]} \right\} = 0, \tag{3.14}$$

and these equations are identically satisfied in consequence of (3.10).

Hence, the integration of (3.1) *is reduced to the algebraic solution of* (3.2) *and the integration of the completely integrable system* (3.13). In the following

sections of this chapter we give some applications to problems of differential geometry.

4. Conformal transformations of a V_n.* The equation

$$'g_{\lambda\mu} = \sigma g_{\lambda\mu} \tag{4.1}$$

gives a conformal transformation of a V_n with fundamental tensor $g_{\lambda\mu}$. The transformed curvature affinor is

$$'K_{\omega\mu\lambda}^{\cdot\cdot\cdot\nu} = K_{\omega\mu\lambda}^{\cdot\cdot\cdot\nu} - g_{[\omega[\lambda}\, s_{\mu]\alpha]}\, g^{\alpha\nu}, \tag{4.2}$$

where

$$s_{\mu\lambda} = 2\nabla_\mu s_\lambda - s_\mu s_\lambda + \tfrac{1}{2} s_\alpha s^\alpha g_{\mu\lambda}, \tag{4.3}$$

$$s_\lambda = \nabla_\lambda \log \sigma. \tag{4.4}$$

The transformation of $K_{\mu\lambda} = K_{\alpha\mu\lambda}^{\cdot\cdot\cdot\alpha}$ is given by

$$'K_{\mu\lambda} = K_{\mu\lambda} + \tfrac{1}{4}\{(n-2)\, s_{\mu\lambda} + s_{\alpha\beta}\, g^{\alpha\beta}\, g_{\mu\lambda}\}. \tag{4.5}$$

If $K_{\mu\lambda}$ vanishes the V_n is often called an "Einstein space". The transformation of the tensor $L_{\mu\lambda}$,

$$\begin{aligned} L_{\mu\lambda} &= -K_{\mu\lambda} + \frac{1}{2(n-1)} K g_{\mu\lambda}; \quad K = K_{\alpha\beta}\, g^{\alpha\beta}, \\ K_{\mu\lambda} &= -L_{\mu\lambda} - \frac{1}{n-2} L g_{\mu\lambda}; \quad L = L_{\alpha\beta}\, g^{\alpha\beta}, \end{aligned} \tag{4.6}$$

is more simple:

$$\frac{4}{n-2}\, 'L_{\mu\lambda} = \frac{4}{n-2} L_{\mu\lambda} - s_{\mu\lambda}. \tag{4.7}$$

It is well known that the V_n can then and only then be transformed conformally into an R_n if the conformal curvature affinor defined by

$$C_{\omega\mu\lambda\nu} = K_{\omega\mu\lambda\nu} - \frac{4}{n-2}\, g_{[\omega[\lambda}\, L_{\mu]\nu]} \tag{4.8}$$

vanishes.†

Now we will deduce the necessary and sufficient conditions in order that a V_n can be transformed conformally into an Einstein space. By (4.7) we have

$$2\nabla_\mu s_\lambda - s_\mu s_\lambda + \tfrac{1}{2} s_\alpha s^\alpha g_{\mu\lambda} = \frac{4}{n-2} L_{\mu\lambda}, \tag{4.9}$$

an equation of the form (1.1).

* Cf. for the formulae (4.1–8), R. K., pp. 168 ff.

† Mathematische Zeitschrift, vol. 11 (1921), p. 83.

The first condition of integrability is

$$
(4.10)\quad
\begin{aligned}
&K_{\omega\mu\lambda}^{\cdot\cdot\cdot\,\nu}\, s_\nu - s_{[\mu}\left\{-\tfrac{1}{4} g_{\omega]\lambda}\, s_\alpha s^\alpha + \frac{2}{n-2} L_{\omega]\lambda}\right\} \\
&+ s^\alpha g_{\lambda[\mu}\left\{\tfrac{1}{2} s_{\omega]}\, s_\alpha - \tfrac{1}{4} g_{\omega]\alpha}\, s_\beta s^\beta + \frac{2}{n-2} L_{\omega]\alpha}\right\} = \frac{4}{n-2} \nabla_{[\omega} L_{\mu]\lambda},
\end{aligned}
$$

or, because of (4.8),

$$
(4.11)\quad C_{\omega\mu\lambda}^{\cdot\cdot\cdot\,\nu}\, s_\nu = \frac{4}{n-2} \nabla_{[\omega} L_{\mu]\lambda}.
$$

By differentiation, using (4.9) to eliminate $\nabla_\mu s_\lambda$ we get the second condition

$$
(4.12)\quad
\begin{aligned}
&\left(\nabla_{[\xi} C_{\omega\mu\lambda]}^{\cdot\cdot\cdot\,\nu}\right) s_\nu + C_{\omega\mu\lambda}^{\cdot\cdot\cdot\,\nu}\left(\tfrac{1}{2} s_\xi\, s_\nu - \tfrac{1}{4} s_\alpha s^\alpha g_{\xi\nu} + \frac{2}{n-2} L_{\xi\nu}\right) \\
&\qquad = \frac{4}{n-2} \nabla_\xi \nabla_{[\omega} L_{\mu]\lambda}.
\end{aligned}
$$

Proceeding in this way we obtain an infinite series of equations. Now these equations may be inconsistent, (4.9) admitting no solution. Or there exists a number q, such that the $(q+1)$th equation is an algebraic consequence of the preceding ones. Then the same holds for all higher equations and (4.9) admits a general solution depending on a finite number of arbitrary constants.

We may also investigate the possibility of conformal transformations leaving $K_{\mu\lambda}$ invariant. Then $L_{\mu\lambda}$ is likewise invariant and (4.9) is reduced to

$$
(4.13)\quad 2\nabla_\mu s_\lambda - s_\mu s_\lambda + \tfrac{1}{2} s_\alpha s^\alpha g_{\mu\lambda} = 0.
$$

Introducing the vector

$$
(4.14)\quad s'_\lambda = s^{-2} s_\lambda; \qquad s^2 = s_\alpha s^\alpha,
$$

(4.13) passes into the simpler equation

$$
(4.15)\quad \nabla_\mu s'_\lambda = -\tfrac{1}{4} g_{\mu\lambda},
$$

of the form (3.1).

The first condition of integrability of (4.15) is

$$
(4.16)\quad K_{\omega\mu\lambda}^{\cdot\cdot\cdot\,\nu}\, s'_\nu = 0.
$$

The other conditions are

$$
(4.17)\quad
\begin{aligned}
&\left(\nabla_{\xi}\ K_{\omega\mu\lambda}^{\cdot\,\cdot\,\cdot\,\nu}\right) s'_{\nu}-\ \ \tfrac{1}{4} K_{\omega\mu\lambda\xi} &&= 0,\\
&\left(\nabla_{\eta\xi}\ K_{\omega\mu\lambda}^{\cdot\,\cdot\,\cdot\,\nu}\right) s'_{\nu}-\tfrac{1}{4}\nabla_{\xi}\ K_{\omega\mu\lambda\eta}-\tfrac{1}{4}\nabla_{\eta}\ K_{\omega\mu\lambda\xi} &&= 0,\\
&\left(\nabla_{\zeta\eta\xi}\ K_{\omega\mu\lambda}^{\cdot\,\cdot\,\cdot\,\nu}\right) s'_{\nu}-\tfrac{1}{4}\nabla_{\eta\xi}\ K_{\omega\mu\lambda\zeta}-\tfrac{1}{4}\nabla_{\zeta\xi}\ K_{\omega\mu\lambda\eta}-\tfrac{1}{4}\nabla_{\zeta\eta}\ K_{\omega\mu\lambda\xi} &&= 0,
\end{aligned}
$$

etc. The conclusions are drawn in the same way as in the earlier case.

We remark that for $n>2$, $g_{[\omega[\lambda}\, s_{\mu]\alpha]}$ vanishes only when $s_{\mu\lambda}=0$. Hence by (4.2) the deduced conditions are for $n>2$ also necessary and sufficient in order that the conformal transformation leaves $K_{\omega\mu\lambda}^{\cdot\,\cdot\,\cdot\,\nu}$ invariant.

5. Geodesic transformations of an A_n.* Every geodesic transformation, viz. a transformation of the $\Gamma_{\lambda\mu}^{\nu}$ leaving invariant the geodesic lines, has the form

$$
(5.1)\qquad {}'\Gamma_{\lambda\mu}^{\nu} = \Gamma_{\lambda\mu}^{\nu} + A_{\lambda}^{\nu}\, p_{\mu} + A_{\mu}^{\nu}\, p_{\lambda},
$$

p_λ being an arbitrary vector, function of x^ν. The transformed curvature affinor is given by

$$
(5.2)\qquad {}'R_{\omega\mu\lambda}^{\cdot\,\cdot\,\cdot\,\nu} = R_{\omega\mu\lambda}^{\cdot\,\cdot\,\cdot\,\nu} - 2\, p_{[\omega\mu]}\, A_{\lambda}^{\nu} + 2\, A_{[\omega}^{\nu}\, p_{\mu]\lambda},
$$

where

$$
(5.3)\qquad p_{\mu\lambda} = \nabla_{\mu}\, p_{\lambda} - p_{\mu}\, p_{\lambda}.
$$

The transformation of $R_{\mu\lambda} = R_{\alpha\mu\lambda}^{\cdot\,\cdot\,\cdot\,\alpha}$ is given by

$$
(5.4)\qquad {}'R_{\mu\lambda} = R_{\mu\lambda} + n\, p_{\mu\lambda} - p_{\lambda\mu}.
$$

The transformation of the tensor $P_{\mu\lambda}$:

$$
(5.5)\qquad
\begin{aligned}
-(n^2-1)\, P_{\mu\lambda} &= n\, R_{\mu\lambda} + R_{\lambda\mu},\\
R_{\mu\lambda} &= -n\, P_{\mu\lambda} + P_{\lambda\mu},
\end{aligned}
$$

is more simple:

$$
(5.6)\qquad {}'P_{\mu\lambda} = P_{\mu\lambda} - p_{\mu\lambda}.
$$

It is well known that the A_n can then and only then be transformed geodesically into an E_n, if the projective curvature affinor, defined by

$$
(5.7)\qquad P_{\omega\mu\lambda}^{\cdot\,\cdot\,\cdot\,\nu} = R_{\omega\mu\lambda}^{\cdot\,\cdot\,\cdot\,\nu} - 2\, P_{[\omega\mu]}\, A_{\lambda}^{\nu} + 2\, A_{[\omega}^{\nu}\, P_{\mu]\lambda},
$$

vanishes.†

* Cf. for the formulae (5.1-7), R. K., pp. 129 ff.

† Weyl, Göttinger Nachrichten (1921), pp. 99–112.

First we will deduce the necessary and sufficient conditions in order that an A_n can be transformed conformally into an A_n with $R_{\mu\lambda} = 0$. By (5.3) and (5.6) we have

$$\nabla_\mu p_\lambda - p_\mu p_\lambda = P_{\mu\lambda}, \tag{5.8}$$

an equation of the form (1.1).

The first condition of integrability is

$$\tfrac{1}{2} R_{\omega\mu\lambda}^{\cdot\cdot\cdot\nu} p_\nu - P_{[\omega\mu]} p_\lambda - p_{[\mu} P_{\omega]\lambda} = \nabla_{[\omega} P_{\mu]\lambda}, \tag{5.9}$$

or, because of (5.7),

$$P_{\omega\mu\lambda}^{\cdot\cdot\cdot\nu} p_\nu = 2 \nabla_{[\omega} P_{\mu]\lambda}. \tag{5.10}$$

By differentiation, using (5.8) to eliminate $\nabla_\mu p_\lambda$, we get the second equation,

$$\left(\nabla_\xi P_{\omega\mu\lambda}^{\cdot\cdot\cdot\nu}\right) p_\nu + P_{\omega\mu\lambda}^{\cdot\cdot\cdot\nu} (p_\xi p_\nu + P_{\xi\nu}) = 2 \nabla_\xi \nabla_{[\omega} P_{\mu]\lambda}. \tag{5.11}$$

Proceeding in this way we obtain an infinite series of equations. The conclusions are drawn in the same way as in the preceding section.

Secondly we consider the possibility of geodesic transformations leaving $R_{\mu\lambda}$ invariant. Then $P_{\mu\lambda}$ is also invariant and (5.8) passes into

$$\nabla_\mu p_\lambda - p_\mu p_\lambda = 0. \tag{5.12}$$

In consequence of this equation p_λ is a gradient vector:

$$p_\lambda = \nabla_\lambda p. \tag{5.13}$$

Introducing the vector

$$p'_\lambda = e^{-p} p_\lambda, \tag{5.14}$$

(5.12) passes into the simpler equation

$$\nabla_\mu p'_\lambda = 0 \tag{5.15}$$

of the form (3.1).

Hence, *a geodesic transformation leaving invariant $R_{\mu\lambda}$ exists only in an A_n admitting a constant vector.* The first condition of integrability of (5.15) is

$$R_{\omega\mu\lambda}^{\cdot\cdot\cdot\nu} p'_\lambda = 0. \tag{5.16}$$

The other conditions are*

$$(5.17)\qquad \begin{aligned} \left(\nabla_\xi\, R_{\omega\mu\lambda}^{\cdot\cdot\cdot\nu}\right) p'_\lambda &= 0,\\ \left(\nabla_{\eta\xi}\, R_{\omega\mu\lambda}^{\cdot\cdot\cdot\nu}\right) p'_\lambda &= 0, \text{ etc.}\end{aligned}$$

The conclusions are drawn in the same way as in the earlier cases. We remark that $p_{[\omega\mu]} A_\lambda^\nu + A^\nu_{[\omega}\, p_{\mu]\lambda}$ vanishes only when $p_{\mu\lambda} = 0$. Hence by (5.2) the deduced conditions are also necessary and sufficient in order that the geodesic tranformation leaves $R_{\omega\mu\lambda}^{\cdot\cdot\cdot\nu}$ invariant.

B. On covariant equations which are linear homogeneous in the first covariant derivatives

6. **Properties of permutational operators.** We consider systems of equations of the form

$$(6.1)\qquad \overset{\xi}{P}\, \nabla_\mu\, v_{\lambda_p} = \overset{\xi}{w}_{\mu\lambda_p} \qquad (\xi = 1, \cdots, z),$$

where the quantity $w_{\mu\lambda_p} = w_{\mu\lambda_p\cdots\lambda_1}$ is a function of x^ν and $v_{\nu_p} = v_{\nu_p\cdots\nu_1}$ only and does not contain derivatives of v_{ν_p}, and where the operators $\overset{\xi}{P}$ are homogeneous linear functions of the $(p+1)!$ permutations of the $p+1$ suffixes $\mu\lambda_p\cdots\lambda_1$ with constant coefficients. So for $p = 2$ a system of this form is, for example,

$$(6.2)\qquad \begin{aligned} a\,\nabla_\mu\, v_{\varkappa\lambda} + b\,\nabla_\varkappa\, v_{\mu\lambda} + c\,\nabla_\lambda\, v_{\varkappa\mu} &= \overset{1}{w}_{\mu\varkappa\lambda},\\ e\,\nabla_\mu\, v_{\lambda\varkappa} + f\,\nabla_\lambda\, v_{\mu\varkappa} &= \overset{2}{w}_{\mu\varkappa\lambda}.\end{aligned}$$

First we have to deduce some purely algebraic properties of the operators P.† These operators form an associative algebra with $(p+1)!$ units, which is composed of k different subalgebras with $\epsilon_1^2, \cdots, \epsilon_k^2$ units, $\epsilon_1^2 + \cdots + \epsilon_k^2 = (p+1)!$; k is the number of different solutions of the equation $x + y + z + \cdots = p + 1$; x, y, z being positive integers. E. g., for $p = 5$, $k = 11$ and the subalgebras have 1, 25, 81, 100, 25, 256, 100, 25, 81, 25, 1 numbers, respectively. The numbers of two different subalgebras annihilate each other by multiplication, and the units $\overset{i}{e}_{uv}$ of any subalgebra with ϵ_i^2 units may be chosen in such a way that

$$(6.3)\qquad \overset{i}{e}_{uv}\,\overset{i}{e}_{wx} = \begin{cases} \overset{i}{e}_{ux}, & v = w\\ 0, & v \neq w \end{cases}, \quad u, v, w, x = 1, \cdots, \epsilon_i.$$

* The conditions (5.17) were first deduced by Eisenhart and Veblen, Proceedings of the National Academy of Sciences, vol. 8 (1922), pp. 19–23.

† Cf. R. K., Abschn. VII.

The number k and the different subalgebras are uniquely determined, and the same holds for the k sums $\sum_u \overset{i}{e}_{uu}$ of the idempotent numbers $\overset{i}{e}_{uu}$. These sums are the moduli of the subalgebras, their sum being the modulus I of the algebra itself. The numbers $\overset{i}{e}_{uu}$ are called *chief-units*. They are not uniquely determined. They form a *chief-series* of the subalgebra and this chief-series can be chosen in an infinite number of ways. A chief-series of the algebra is built up by k chief-series of the k subalgebras, each chief-series chosen in an arbitrary way.

The rules of multiplication of the numbers of a subalgebra are the same as the rules of first transvection (erste Überschiebung) of the mixed affinors of the second degree in ε_i variables. Hence the operator P can be written*

$$\sum_i P_{\alpha_i}^{\cdot\,\gamma_i}; \qquad \alpha_i, \gamma_i = 1, \cdots, \varepsilon_i; \qquad i = 1, \cdots, k, \tag{6.4}$$

or, briefly, $P_\alpha^{\cdot\,\gamma}$, where α and γ take only values from $1, \cdots, N$, $N = \sum_i \varepsilon_i$, belonging to the same subalgebra. With this notation the properties of the operators P can be easily deduced from the well known properties of the affinors $P_\alpha^{\cdot\,\gamma}$ in a point of an E_N.

N linearly independent contravariant vectors $\underset{\alpha}{e}^\gamma$ of the E_N being given, every affinor $P_\alpha^{\cdot\,\gamma}$ can be written in one and only one way in the form

$$P_\alpha^{\cdot\,\gamma} = \overset{\beta}{u}_\alpha \underset{\beta}{e}^\gamma. \tag{6.5}$$

From the vectors $\overset{\gamma}{u}_\alpha$ there will be some, say r, linearly independent. If we denote them by $\overset{u}{p}_\alpha$, $u = 1, \cdots, r$, there exist r vectors $\underset{u}{q}^\gamma$, such that

$$P_\alpha^{\cdot\,\gamma} = \sum_u^{1,\cdots,r} \overset{u}{p}_\alpha \underset{u}{q}^\gamma. \tag{6.6}$$

r is the *rank* of P; this number is the sum of the *subranks* $r_1, \cdots, r_k$, belonging to the k subalgebras. $r_1, \cdots, r_k$ are invariants of P. So are the E_r and the E_{n-r} determined by the alternate products $\underset{1}{q}^{[\gamma_1} \cdots \underset{r}{q}^{\gamma_r]}$ and $\overset{1}{p}_{[\alpha_1} \cdots \overset{r}{p}_{\alpha_r]}$, called the *post-region* and the *pre-region* of P. The vectors p_α and q^γ are not uniquely determined, but the set p_α is, when the set q^γ is given in E_r, and vice versa.

---

* In this formula α_i and γ_i do *not* stand for $\alpha_i \cdots \alpha_1$ and $\gamma_i \cdots \gamma_1$.

The operator P is then and only then idempotent when, for some choice of the set p_α, q^γ,

$$\overset{u}{p}_\alpha \underset{v}{q}^\alpha = \begin{cases} 1, & u = v \\ 0, & u \neq v \end{cases}; \qquad u, v = 1, \cdots, r. \tag{6.7}$$

Then (6.7) holds for every choice of p_α or q^γ. To every idempotent operator E belongs its conjugate operator $E = I - E$, which is also idempotent. The pre-region of E is the post-region of E' and vice versa. Pre- and post-regions of an idempotent operator have no directions in common. P being idempotent, $\overset{u}{p}_\alpha \underset{u}{q}^\gamma$ is a chief-unit for every u, hence an idempotent operator of rank r is a sum of r chief-units, r_1 from the first subalgebra, r_2 from the second, etc. The chief-units themselves are not uniquely determined, but the sums of the chief-units belonging to the same subalgebra are.

To every operator P belongs an infinite number of idempotent operators having the same pre-(post-)region as P. Having written P in the form (6.6), we have only to choose in an arbitrary way $N - r$ covariant vectors $\overset{r+1}{p}_\alpha, \cdots, \overset{N}{p}_\alpha$ linearly independent of each other and of $\overset{1}{p}_\alpha, \cdots, \overset{r}{p}_\alpha$, and form the reciprocal system $\underset{1}{p}^\gamma, \cdots, \underset{N}{p}^\gamma$, so that

$$\underset{u}{p}^\alpha \overset{v}{p}_\alpha = \begin{cases} 1, & u = v \\ 0, & u \neq v \end{cases}. \tag{6.8}$$

Then

$$E_\alpha^{\cdot\gamma} = \sum_u^{1,\cdots,r} \overset{u}{p}_\alpha \underset{u}{p}^\gamma \tag{6.9}$$

is an idempotent operator with the same pre-region as P. In the same way we form an idempotent operator F, having the same post-region as P, using $N - r$ vectors $\underset{r+1}{q}^\gamma, \cdots, \underset{N}{q}^\gamma$ and the reciprocal system $\overset{1}{q}_\alpha, \cdots, \overset{N}{q}_\alpha$. From the definition of E and F we have

$$EP = P; \qquad PF = P, \tag{6.10}$$

and

$$\begin{aligned} E_\alpha^{\cdot\gamma} &= \sum_u^{1,\cdots,N} P_\alpha^{\cdot\beta} \overset{u}{q}_\beta \underset{u}{p}^\gamma, \\ F_\alpha^{\cdot\gamma} &= \sum_u^{1,\cdots,N} \overset{u}{q}_\alpha \underset{u}{p}^\beta P_\beta^{\cdot\gamma}. \end{aligned} \tag{6.11}$$

Hence, E and F can be obtained from P by post-(pre-)multiplication *with an operator of rank N.*

Now we are able to consider the question when it is possible to write a quantity $w_{\mu\lambda_p}$ in the form

$$w_{\mu\lambda_p} = P u_{\mu\lambda_p}. \tag{6.12}$$

Operating on (6.12) with E we have, by (6.10),

$$E w_{\mu\lambda_p} = w_{\mu\lambda_p}. \tag{6.13}$$

An equivalent to (6.13) is

$$E' w_{\mu\lambda_p} = 0, \tag{6.14}$$

E' being the conjugate to E. But, by using the reciprocal system $\overset{1}{q}_\alpha, \cdots, \overset{N}{q}_\alpha$ we have, in consequence of (6.13) and (6.11),

$$\begin{aligned} w_{\mu\lambda_p} &= E_\alpha^{\cdot\gamma}\, w_{\mu\lambda_p} \\ &= P_\alpha^{\cdot\beta} \sum_u^{1,\cdots,N} \overset{u}{q}_\beta\, \underset{u}{p}^\gamma\, w_{\mu\lambda_p}. \end{aligned} \tag{6.15}$$

Thus (6.13) *or* (6.14) *is a necessary and sufficient condition that* $w_{\mu\lambda_p}$ *be a result of the operation* P.

Hence, if $\overset{x}{E}$ is an idempotent operator, having the same pre-region as $\overset{x}{P}$ in (6.1), and $\overset{x}{E}'$ is the conjugate operator,

$$\overset{\xi}{E}'\, \overset{\xi}{w}_{\mu\lambda_p} = 0 \qquad (\xi = 1, \cdots, z) \tag{6.16}$$

are *the first conditions of integrability* of the system (6.1).

We now prove the following theorem:

A consistent system of equations of the form (6.1) *is equivalent to only one equation of that form.**

We consider the first two equations of (6.1). If the ranks of $\overset{1}{P}$ and $\overset{2}{P}$ are $\overset{1}{r}$ and $\overset{2}{r}$, and if the post-regions of these operators have s directions in common, these equations can be written in the form

$$\begin{gathered} \sum_1^{\overset{1}{r}}{}^u\, \overset{u}{p}_\alpha\, \underset{u}{q}^\gamma\, \nabla_\mu v_{\lambda_p} = \overset{1}{w}_{\mu\lambda_p}, \\ \sum_{y+1}^{t}{}^u\, \overset{u}{r}_\alpha\, \underset{u}{q}^\gamma\, \nabla_\mu v_{\lambda_p} = \overset{2}{w}_{\mu\lambda_p}, \\ \overset{1}{r} - y = s, \qquad t - y = \overset{2}{r}. \end{gathered} \tag{6.17}$$

* The algebraic equivalent of this theorem was first deduced by Weyl, Rendiconti del Circolo Matematico di Palermo, vol. 48 (1924), p. 32.

Completing the sets p_α, q^γ and r_α in an arbitrary way to sets of N linearly independent vectors and forming the reciprocal sets p^γ, q_α and r^γ, (6.17) is equivalent to

$$
\begin{aligned}
&\sum_{1}^{\overset{1}{r}}{}^{u}\, \overset{u}{q}_\alpha\, \underset{u}{q}^{\gamma}\, \nabla_\mu v_{\lambda_p} = \sum_{1}^{N}{}^{u}\, \overset{u}{q}_\alpha\, \underset{u}{p}^{\gamma}\, \overset{1}{w}_{\mu\lambda_p},\\
(6.18)\qquad &\sum_{y+1}^{t}{}^{u}\, \overset{u}{q}_\alpha\, \underset{u}{q}^{\gamma}\, \nabla_\mu v_{\lambda_p} = \sum_{1}^{N}{}^{u}\, \overset{u}{q}_\alpha\, \underset{u}{r}^{\gamma}\, \overset{2}{w}_{\mu\lambda_p}.
\end{aligned}
$$

In order that these equations be consistent it is necessary and sufficient that

$$
\begin{aligned}
&\overset{u}{q}_\alpha\, \underset{u}{p}^{\gamma}\, \overset{1}{w}_{\mu\lambda_p} = \overset{u}{q}_\alpha\, \underset{u}{r}^{\gamma}\, \overset{2}{w}_{\mu\lambda_p} \qquad (u = y+1, \cdots, \overset{1}{r});\\
(6.19)\qquad &\overset{u}{q}_\alpha\, \underset{u}{p}^{\gamma}\, \overset{1}{w}_{\mu\lambda_p} = 0 \qquad (u = \overset{1}{r}+1, \cdots, N);\\
&\overset{u}{q}_\alpha\, \underset{u}{r}^{\gamma}\, \overset{2}{w}_{\mu\lambda_p} = 0 \qquad (u = 1, \cdots, y,\ t+1, \cdots, N).
\end{aligned}
$$

In the case of consistency (6.18) is equivalent to

$$
\begin{aligned}
&\overset{u}{q}_\alpha\, \underset{u}{p}^{\gamma}\, \nabla_\mu v_{\lambda_p} = \overset{u}{q}_\alpha\, \underset{u}{p}^{\gamma}\, \overset{1}{w}_{\mu\lambda_p} \qquad (u = 1, \cdots, \overset{1}{r});\\
(6.20)\qquad &\overset{u}{q}_\alpha\, \underset{u}{p}^{\gamma}\, \nabla_\mu v_{\lambda_p} = \overset{u}{q}_\alpha\, \underset{u}{r}^{\gamma}\, \overset{2}{w}_{\mu\lambda_p} \qquad (u = \overset{1}{r}+1, \cdots, t),
\end{aligned}
$$

or

$$
(6.21)\qquad \sum_{1}^{t}{}^{u}\, \overset{u}{q}_\alpha\, \underset{u}{q}^{\gamma}\, \nabla_\mu v_{\lambda_p} = \sum_{1}^{\overset{1}{r}}{}^{u}\, \overset{u}{q}_\alpha\, \underset{u}{p}^{\gamma}\, \overset{1}{w}_{\mu\lambda_p} + \sum_{\overset{1}{r}+1}^{t}{}^{u}\, \overset{u}{q}_\alpha\, \underset{u}{r}^{\gamma}\, \overset{2}{w}_{\mu\lambda_p},
$$

q. e. d. Proceeding in this way it can be proved that all equations (6.1) are equivalent to only one equation.

It may be remarked that the post-region of the operator in the left side of (6.21) is the smallest region that contains the post-regions of both $\overset{1}{P}$ and $\overset{2}{P}$. In the same way the operator in the left side of the equation equivalent to all equations (6.1) has a post-region that is the smallest one containing all post-regions of the operators P. We call this region the combined post-region of the operators P. The sum of the dimensions of the combined pre-region, constructed in the same way, and the combined post-region is in general not equal to N.

Suppose now that an algebraic equation is given of the form

$$
(6.22)\qquad P u_\mu\, v_{\lambda_p} = w_{\mu\lambda_p},
$$

u_μ and $w_{\mu\lambda_p}$ being known quantities. The left side of this equation can in one and only one manner be written in the form

$$(6.23)\qquad P u_\mu v_{\lambda_p} = \overset{1}{\varphi} u_\mu v_{\lambda_p} + \overset{2}{C}\overset{2}{\varphi} u_\mu v_{\lambda_p} + \cdots + \overset{p+1}{C}\overset{p+1}{\varphi} u_\mu v_{\lambda_p},$$

where $\overset{1}{\varphi}, \cdots, \overset{p+1}{\varphi}$ operate only on $\lambda_1, \cdots, \lambda_p$ and $\overset{2}{C}, \cdots, \overset{p+1}{C}$ are the cyclical operators transforming $\mu\lambda_1 \cdots \lambda_p$ into $\lambda_1 \cdots \lambda_p\mu$, $\lambda_2 \cdots \lambda_p \mu \lambda_1$, etc. The operators operating only on $\lambda_1 \cdots \lambda_p$ form an associative algebra with $p!$ units. In this algebra the operators $\overset{1}{\varphi}, \cdots, \overset{p+1}{\varphi}$ have a combined post-region and we can form an idempotent operator U having this same post-region. Let U' be the conjugate operator of U. Then x_{λ_p} being any quantity of degree p, we have identically

$$(6.24)\qquad P u_\mu U' x_{\lambda_p} = 0.$$

On the other hand every solution of $P u_\mu v_{\lambda_p} = 0$ has the form $U' x_{\lambda_p}$. Therefore, we have only to consider such solutions of (6.22) as satisfy the equation

$$(6.25)\qquad U v_{\lambda_p} = v_{\lambda_p}.$$

These solutions being known, every other solution can be found by adding $U' x_{\lambda_p}$, x_{λ_p} being an arbitrary quantity of degree p.

7. **Determination of the characteristic number of a permutational operator.** In the preceding section we have seen that every operator has a definite *rank*. Now we will determine the number of linearly independent components of a quantity $P v_{\lambda_q}$, v_{λ_q} being an arbitrary quantity of degree q and P a homogeneous linear function of the $q!$ permutations of $\lambda_1 \cdots \lambda_q$. We call this number the *characteristic number* of P and denote it by (P). In some simple cases the characteristic number can easily be found.* If for instance P alternates (mixes)† the first t suffixes, leaving the other suffixes unchanged, we have obviously

$$(7.1)\qquad (P) = \binom{n}{t} n^{q-t}$$

and

$$(7.2)\qquad (P) = \binom{n+t-1}{t} n^{q-t}$$

* Comp. p. 459, note *.

† By alternating we get for instance from $v_{\varkappa\lambda\mu}$

$$v_{[\varkappa\lambda\mu]} = \frac{1}{3!}(v_{\varkappa\lambda\mu} + v_{\lambda\mu\varkappa} + v_{\mu\varkappa\lambda} - v_{\varkappa\mu\lambda} - v_{\lambda\varkappa\mu} - v_{\mu\lambda\varkappa})$$

and by mixing

$$v_{(\varkappa\lambda\mu)} = \frac{1}{3!}(v_{\varkappa\lambda\mu} + v_{\lambda\mu\varkappa} + v_{\mu\varkappa\lambda} + v_{\varkappa\mu\lambda} + v_{\lambda\varkappa\mu} + v_{\mu\lambda\varkappa}).$$

respectively. In the general case it would be theoretically possible to determine the non-vanishing determinants of the matrix of the components of Pv_{λ_q}. In practice this method is far too laborious. But we can find the desired number in a very easy way, making use of the author's development of a covariant quantity in a series of indivisible quantities, viz. quantities that have no linear covariant with a smaller number of components. These indivisible quantities correspond to the chief-units of the algebra of permutational operators, and in consequence it is possible to write any quantity as a sum of indivisible quantities, using an arbitrary chosen chief series $\overset{1}{I}, \cdots, \overset{n}{I}$. By using this chief-series we obtain the development

$$Pv_{\lambda_q} = (\overset{1}{I} + \cdots + \overset{N}{I})\, Pv_{\lambda_q}. \tag{7.3}$$

Now among the N indivisible quantities, obtained in this way, some may be zero, and others may be each other's covariants. If r is the rank of P, just r of them are not zero and independent. The characteristic number of one of the remaining r operators $\overset{\alpha}{I}P$ is the same as the characteristic number of $\overset{\alpha}{I}$, because $\overset{\alpha}{I}v_{\lambda_q}$ has no linear covariants with a smaller number of components. But $(\overset{\alpha}{I})$ can easily be found, because it is possible to choose the chief-series in a very simple way.* The process is shortened by the circumstance that all chief-units, belonging to the same subalgebra, have the same characteristic number. So we have only to determine the subranks $\overset{1}{r}, \cdots, \overset{k}{r}$ and the characteristic numbers belonging to each of the k subalgebras.

The method just indicated can be considered from another point of view. The equation

$$Pv_{\lambda_q} = 0 \tag{7.4}$$

being equivalent to

$$Fv_{\lambda_q} = 0, \tag{7.5}$$

where F is an idempotent operator with the same post-region as P, (P) and (F) must be equal. Now F can be written (p. 455) as a sum of chief-units,† and so we have only to determine the characteristic numbers of these chief-units. Each of the indivisible quantities belonging to these chief-units is a linear covariant of one of the terms $\overset{\alpha}{I}Pv_{\lambda_q}$ obtained above, but the

* Also a general formula can be used deduced by I. Schur in an investigation on the theory of the linear homogeneous group (Dissert. 1901, p. 51). Comp. H. Weyl, Mathematische Zeitschrift, vol. 23 (1925), p. 300.

† These chief-units were first used by Weyl in an investigation concerning the theory of invariants, Rendiconti del Circolo Matematico di Palermo, vol. 48 (1924), pp. 29–36.

chief-units contained in F do not belong in general to the artificial chief-series we made use of. This way seems to be a shorter one, but in practice it is not so easy to determine the r chief-units belonging to F and so the use of the artificial chief-series must be recommended.

We give some examples, which will be used later on.

Example 1. Let ${}_xA$ be an operator, alternating the first x suffixes $\lambda_q \cdots \lambda_{q-x+1}$ and ${}_{y+1}M$, $x+y=q$, an operator mixing λ_q and the last $q-x$ suffixes $\lambda_{q-x}, \cdots, \lambda_1$. Then a number ${}_xA\,{}_{y+1}M$ is a chief-unit of an artificial chief-series as used above. Also ${}_{y+1}M\,{}_xA$ is chief-unit of another artificial chief-series, frequently used. For $q<6$ all chief-units of artificial chief-series have such a form, though there may be more than one region for alternating or mixing.* The characteristic number of ${}_xA\,{}_{y+1}M$ and ${}_{y+1}M\,{}_xA$ is

$$
\begin{aligned}
(7.6)\qquad &({}_xA\,{}_{y+1}M) = ({}_{y+1}M\,{}_xA) \\
&= \binom{n}{x-1}\binom{n+y}{y+1} - \binom{n}{x-2}\binom{n+y+1}{y+2} + \cdots + (-1)^{x-1}\binom{n+q-1}{q}.
\end{aligned}
$$

The first term in this series is $({}_{x-1}A\,{}_{y+1}M)$, ${}_{x-1}A$ alternating $x-1$ suffixes and ${}_{y+1}M$ mixing the other $y+1$, the second is $({}_{x-2}A\,{}_{y+2}M)$ etc.

Example 2. Let ${}_xM$ be an operator mixing the first x suffixes $\lambda_q \cdots \lambda_{q-x+1}$ and ${}_{y+1}M$ the operator used in Example 1. By using the above mentioned second chief-series we find that $({}_xM\,{}_{y+1}M)$ is equal to $({}_xM\,{}_yM)$, ${}_yM$ mixing only $\lambda_{x+1}, \cdots, \lambda_y$. Hence we have

$$
(7.7)\qquad ({}_xM\,{}_{y+1}M) = ({}_xM\,{}_yM) = \binom{n+x-1}{x}\binom{n+y-1}{y}.
$$

8. **A theorem of Lie.** Given a system of partial differential equations with n independent variables x^ν and an arbitrary number of dependent ones. There are three cases:

1. The equations are *inconsistent.*
2. The general solution depends on a *finite* number of arbitrary constants.
3. The general solution contains *arbitrary functions* of x^ν.

An example of the first and second case is the equation

$$
(8.1)\qquad \nabla_\mu v_\lambda = 0,
$$

having in general no solution and in special cases a general solution depending on a finite number of arbitrary constants. An example of the third case is the equation

$$
(8.2)\qquad \nabla_{[\mu} v_{\lambda]} = 0,
$$

* For $n \geq 6$ at least one chief-unit in every subalgebra has such a form.

having the general solution

$$v_\lambda = \nabla_\lambda p, \tag{8.3}$$

p being an arbitrary scalar function.

The equations being consistent, Lie* has proved that the following conditions are necessary and sufficient for the second case:

1. There exists a number s, such that every derivative of order s may be expressed as a function of the lower derivatives, and that the same does not hold for every derivative of order $s-1$.

2. By differentiating and eliminating the system can be put in such a form that, by differentiating all equations once, and eliminating all derivatives of order $s+1$, no equation can be obtained that is not a consequence of the equations of the system.

Let us inquire in which cases the first of Lie's conditions holds for an equation of the form

$$P \nabla_\mu v_{\lambda_p} = w_{\mu\lambda_p}, \tag{8.4}$$

where $w_{\mu\lambda_p}$ is a function of x^ν and v_{λ_p} only, not containing derivatives of v_{λ_p}. If U is an idempotent operator in $\lambda_1 \cdots \lambda_p$ only, belonging to P as defined in § 6, U' the conjugate of U, $\overset{0}{v}_{\lambda_p}$ any solution of (8.4), and s_{λ_p} an arbitrary quantity of degree p, then

$$\overset{0}{v}_{\lambda_p} + U' s_{\lambda_p} \tag{8.5}$$

is also a solution of (8.4), because P annihilates U'. Hence the general solution of (8.4) depends for $U \neq I$ not on a finite number of arbitrary constants, when no other assumption is made. In consequence we consider only solutions of (8.4) satisfying the equation

$$U v_{\lambda_p} = v_{\lambda_p}. \tag{8.6}$$

According to § 6 the first condition of integrability of (8.4) is

$$E' w_{\mu\lambda_p} = 0, \tag{8.7}$$

E' being the conjugate of an idempotent operator having the same pre-region as P. This condition being satisfied, (8.4) can then and only then be solved for $\nabla_\mu v_{\lambda_p}$ when the number of components of the left side is equal to $n(U)$. Now this number is equal to (P). Hence, if $(P) = n(U)$,

* *Theorie der Transformationsgruppen*, I, Chapter 10.

(8.4) can be put into a form already considered in Chapter I. Supposing now $(P) < n(U)$ and differentiating (8.4) we have

$$P \nabla_{\mu_2} v_{\lambda_p} = \nabla_{\mu_2} w_{\mu_1 \lambda_p}. \tag{8.8}$$

The number of components of $P \nabla_{\mu_2} v_{\lambda_p}$ and $\nabla_{\mu_2} v_{\lambda_p}$ being $n(P)$ and $n^2(U)$, (8.8) cannot be solved for $\nabla_{\mu_2} v_{\lambda_p}$. From (8.8) we obtain, however,

$$P \nabla_{(\mu_2)} v_{\lambda_p} = \nabla_{\mu_2} w_{\mu_1 \lambda_p} - \tfrac{1}{2} P R_{\mu_2 \lambda_p}^{\cdot\,\cdot\;\nu_p} v_{\nu_p}, \tag{8.9}$$

and in this equation the number of components of $P \nabla_{(\mu_2)} v_{\lambda_p}$ is equal to $(P\,{}_2M)$, ${}_2M$ being the operator mixing the suffixes $\mu_2\, \mu_1$. The number of components of $\nabla_{(\mu_2)} v_{\lambda_p}$ being $({}_2M\,U)$, the solution of (8.9) for $\nabla_{(\mu_2)} v_{\lambda_p}$ is then and only then possible when $(P\,{}_2M) = ({}_2M\,U)$. In this case the right side of (8.9) containing only v_{λ_p} and first derivatives, $\nabla_{(\mu_2)} v_{\lambda_p}$ and therefore also $\nabla_{\mu_2} v_{\lambda_p}$ can be expressed as a function of v_{λ_p} and first derivatives, so that the first of Lie's conditions is satisfied for $s = 2$. When $(P\,{}_2M) < ({}_2M\,U)$ we have, by differentiating (8.8),

$$P \nabla_{\mu_3} v_{\lambda_p} = \nabla_{\mu_3} w_{\mu_1 \lambda_p}. \tag{8.10}$$

Now, because of the identity

$$\begin{aligned} p_{\mu_3 \mu_2 \mu_1} = p_{(\mu_3 \mu_2 \mu_1)} &- \tfrac{5}{8} p_{[\mu_2 \mu_3]\mu_1} + \tfrac{1}{2} p_{\mu_2[\mu_3 \mu_1]} \\ &+ \tfrac{1}{8} p_{[\mu_2 \mu_1]\mu_3} - \tfrac{5}{8} p_{\mu_3[\mu_1 \mu_2]} \\ &- \tfrac{1}{2} p_{\mu_1[\mu_2 \mu_3]} + \tfrac{1}{8} p_{[\mu_3 \mu_1]\mu_2} {}^{*} \end{aligned} \tag{8.11}$$

every quantity p_{μ_3} can be written as a sum of $p_{(\mu_3)}$ and several quantities all alternating in two *neighboring* suffixes. Making use of this property and bringing all terms alternating in two neighboring suffixes to the right, reducing them to first derivatives by means of $R_{\omega\mu\lambda}^{\cdot\,\cdot\,\cdot\;\nu}$, we obtain an equation of the form

$$\begin{aligned} P\,{}_3M \nabla_{\mu_3} v_{\lambda_p} = \nabla_{\mu_3} w_{\mu_1 \lambda_p} &+ \overset{3}{T}{}_{\mu_3 \lambda_p}^{\cdot\,\cdot\;\nu_p} v_{\nu_p} \\ &+ \overset{3}{T}{}_{\mu_3 \lambda_p}^{\cdot\,\cdot\;\alpha_1 \nu_p} \nabla_{\alpha_1} v_{\nu_p} \end{aligned} \tag{8.12}$$

containing only derivatives up to the second on the right. In this way we can proceed, always reducing the difference of $\nabla_{\mu_z} v_{\lambda_p}$ and $\nabla_{(\mu_z)} v_{\lambda_p}$

* In this equation the abridged notation is not used.

to derivatives of the order $x-2$ with the aid of $R_{\omega\mu\lambda}^{\cdot\cdot\cdot\nu}$. Then it may occur that for $x = q$, $(P_q M) = ({}_q M U)$. Then $\nabla_{(\mu_q)} v_{\lambda_p}$ and therefore $\nabla_{\mu_q} v_{\lambda_p}$ can be expressed as functions of the lower derivatives, so that the first of Lie's conditions is satisfied for $s = q$. In the other case such an expression can never be found and in the case of consistency (8.4) must have a general solution containing arbitrary functions.

We give some examples.

Example 1.

$$\nabla_{[\mu} v_{\lambda_t]\lambda_p} = w_{\mu\lambda_p}; \qquad t \leq p; \tag{8.13}$$

P is the operator alternating the suffixes $\mu\lambda_1 \cdots \lambda_t$ and U is the operator alternating $\lambda_1 \cdots \lambda_t$. The characteristic numbers are, according to § 7,

$$\begin{aligned}(P_x M) &= n^{p-t}\left\{\binom{n}{t}\binom{n+x-1}{x}-\binom{n}{t-1}\binom{n+x}{x+1}+\cdots+(-1)^t\binom{n+x+t-1}{t+x}\right\},\\ ({}_x M U) &= n^{p-t}\left\{\binom{n}{t}\binom{n+x-1}{x}\right\},\end{aligned} \tag{8.14}$$

so that always

$$(P_x M) < ({}_x M U). \tag{8.15}$$

Hence, (8.13) is inconsistent or the general solution contains arbitrary functions.

Example 2.

$$\nabla_{(\mu} v_{\lambda_t)\lambda_p} = w_{\mu\lambda_p}; \qquad t \leq p; \tag{8.16}$$

P is the operator mixing the suffixes $\mu\lambda_1 \cdots \lambda_t$ and U mixes $\lambda_1 \cdots \lambda_t$. The characteristic numbers are, according to § 7,

$$\begin{aligned}(P_x M) &= n^{p-t}\binom{n+t}{t+1}\binom{n+x-2}{x-1},\\ ({}_x M U) &= n^{p-t}\binom{n+t-1}{t}\binom{n+x-1}{x}.\end{aligned} \tag{8.17}$$

Whence

$$\begin{aligned}(P_x M) &< ({}_x M U), \qquad & x &\leq t,\\ (P_x M) &= ({}_x M U), \qquad & x &= t+1.\end{aligned} \tag{8.18}$$

Hence, the derivatives of order $t+1$ can be expressed as functions of the derivatives of lower order, and therefore the first of Lie's conditions is satisfied for $s = t+1$.

9. **The second of Lie's conditions for covariant equations.** Considering the case that the first of Lie's conditions is satisfied, we start with (8.4), (8.9), (8.10) and the higher equations, deduced as explained in the preceding section:

$$P \nabla_{\mu} v_{\lambda_p} = w_{\mu\lambda_p}; \tag{9.1}$$

$$\begin{aligned} P_x M \nabla_{\mu_x} v_{\lambda_p} = \nabla_{\mu_x} w_{\mu_1\lambda_p} &+ \overset{x}{T}{}^{\cdot\,\cdot\,\nu_p}_{\mu_x\lambda_p} v_{\nu_p} \\ &+ \overset{x}{T}{}^{\cdot\,\cdot\,\alpha_1\nu_p}_{\mu_x\lambda_p} \nabla_{\alpha_1} v_{\nu_p} + \cdots \\ &+ \overset{x}{T}{}^{\cdot\,\cdot\,\alpha_{x-1}\nu_p}_{\mu_x\lambda_p} \nabla_{\alpha_{x-1}} v_{\nu_p} \\ &\qquad (x = 2, \cdots, q-1); \end{aligned} \tag{9.2}$$

$$\begin{aligned} P_q M \nabla_{\mu_q} v_{\lambda_p} = \nabla_{\mu_q} w_{\mu_1\lambda_p} &+ \overset{q}{T}{}^{\cdot\,\cdot\,\nu_p}_{\mu_q\lambda_p} v_{\nu_p} \\ &+ \overset{q}{T}{}^{\cdot\,\cdot\,\alpha_1\nu_p}_{\mu_q\lambda_p} \nabla_{\alpha_1} v_{\nu_p} + \cdots \\ &+ \overset{q}{T}{}^{\cdot\,\cdot\,\alpha_{q-1}\nu_p}_{\mu_q\lambda_p} \nabla_{\alpha_{q-1}} v_{\nu_p}. \end{aligned} \tag{9.3}$$

The last equation, containing on the right side only derivatives up to the $(q-1)$th, being solved with respect to $\nabla_{(\mu_q)} v_{\lambda_p}$, we arrive at an equation of the form

$$\nabla_{\mu_q} v_{\lambda_p} = Q_{\mu_q\lambda_p}, \tag{9.4}$$

$Q_{\mu_q\lambda_p}$ containing only derivatives up to the $(q-1)$th. By differentiating (9.4) and alternating we have

$$\tfrac{1}{2} R_{\omega\mu_q\lambda_p}^{\cdot\,\cdot\,\cdot\,\alpha_{q-1}\nu_p} \nabla_{\alpha_{q-1}} v_{\lambda_p} = \nabla_{[\omega} Q_{\mu_q]\mu_{q-1}\lambda_p}. \tag{9.5}$$

The qth derivatives on the right being eliminated using (9.4), this equation contains only derivatives up to the $(q-1)$th. Hence it can be no algebraic consequence of (9.4). Differentiating (9.5) and eliminating always the qth derivatives by means of (9.4) leads to an infinite series of equations containing derivatives only up to the $(q-1)$th. Now either (9.1), (9.2), (9.5) and the equation obtained in this way are algebraically inconsistent, or they are algebraic consequences of a finite number of them. In the latter case we know by a general theorem proved by Delassus* that if one of the equations deduced from (9.5) by differentiation and elimination is an algebraic consequence of the preceding ones and (9.1), (9.2) and (9.5),

* Annales de l'Ecole Normale Supérieure, vol. 13 (1896), p. 449.

then all following equations have the same property. Hence it is not necessary to consider these equations. In this latter case the second of Lie's conditions is satisfied because there is no other way of differentiating and eliminating of the sth derivatives than the way just shown.

Summing up, we have the following theorem:

A system of covariant equations of the form

$$\overset{\xi}{P} \nabla_\mu v_{\lambda_p} = \overset{\xi}{w}_{\mu\lambda_p}, \qquad (\xi = 1, \cdots, z)$$

is equivalent to one equation of that form,

$$P \nabla_\mu v_{\lambda_p} = w_{\mu\lambda_p}.$$

The equation being consistent and U being an idempotent operator in $\lambda_1 \cdots \lambda_p$ belonging to P, the general solution depends then and only then on a finite number of arbitrary constants, when there exists a number q such that the characteristic numbers of the operators $P_q M$ and ${}_q M U$ are equal:

$$(P_q M) = ({}_q M U).$$

If for every x

$$(P_x M) < ({}_x M U),$$

the general solution contains arbitrary functions of x^ν.

10. **The linear case.** When $w_{\mu\lambda_p}$ is a linear function of v_{λ_p} the equations (9.1) and (9.2) take the form*

$$P \nabla_\mu v_{\lambda_p} = \overset{0}{u}_{\mu\lambda_p} + \overset{1}{u}{}_{\mu\lambda_p}^{\cdot\cdot\,\nu_p} v_{\nu_p}, \tag{10.1}$$

$$\begin{aligned} P_x M \nabla_{\mu_x} v_{\lambda_p} = \nabla_{\mu_x} \overset{0}{u}_{\mu_1\lambda_p} &+ \overset{x}{u}{}_{\mu_x\lambda_p}^{\cdot\cdot\,\nu_p} v_{\nu_p} \\ &+ \overset{x}{u}{}_{\mu_x\lambda_p}^{\cdot\cdot\,\alpha_1\nu_p} \nabla_{\alpha_1} v_{\nu_p} + \cdots \\ &+ \overset{x}{u}{}_{\mu_x\lambda_p}^{\cdot\cdot\,\alpha_{x-1}\nu_p} \nabla_{\alpha_{x-1}} v_{\nu_p} \end{aligned} \tag{10.2}$$

$$(x = 2, \cdots, q-1);$$

* The case in which the right side of (10.1) is zero, $p = 2$, and P is the operator mixing the suffixes $\mu\lambda_1\lambda_2$, is treated by Veblen and Thomas, these Transactions, vol. 25 (1923), pp. 599 ff.

$$
\begin{aligned}
P_q M \nabla_{\mu_q} v_{\lambda_p} = \nabla_{\mu_q} \overset{0}{u}_{\mu_1\lambda_p} &+ \overset{q}{u}{}_{\mu_q\lambda_p}^{\cdot\,\cdot\,\nu_p} v_{\nu_p} \\
&+ \overset{q}{u}{}_{\mu_q\lambda_p}^{\cdot\,\cdot\,\alpha_1\nu_p} \nabla_{\alpha_1} v_{\nu_p} + \cdots \\
&+ \overset{q}{u}{}_{\mu_q\lambda_p}^{\cdot\,\cdot\,\alpha_{q-1}\nu_p} \nabla_{\alpha_{q-1}} v_{\nu_p};
\end{aligned}
\tag{10.3}
$$

where all quantities u are functions of x^ν only, not containing v_{λ_p} or derivatives of v_{λ_p}. The last equation contains only derivatives up to the $(q-1)$th on the right. The solution of the last equation for ${}_qM \nabla_{\mu_q} v_{\lambda_p}$ gives rise to an equation of the form

$$
\begin{aligned}
\nabla_{\mu_q} v_{\lambda_p} = \overset{0}{U}_{\mu_q\lambda_p} &+ U_{\mu_q\lambda_p}^{\cdot\,\cdot\,\nu_p} v_{\nu_p} \\
&+ U_{\mu_q\lambda_p}^{\cdot\,\cdot\,\alpha_1\nu_p} \nabla_{\alpha_1} v_{\nu_p} + \cdots \\
&+ U_{\mu_q\lambda_p}^{\cdot\,\cdot\,\alpha_{q-1}\nu_p} \nabla_{\alpha_{q-1}} v_{\nu_p},
\end{aligned}
\tag{10.4}
$$

containing only derivatives up to the $(q-1)$th on the right. By differentiation and alternation of (10.4) and elimination of the qth derivatives by means of (10.4) we get an equation of the form

$$
\begin{aligned}
\overset{01}{V}_{\omega\mu_q\lambda_p} + \overset{1}{V}{}_{\omega\mu_q\lambda_p}^{\cdot\,\cdot\,\cdot\,\nu_p} v_{\nu_p} + \overset{1}{V}{}_{\omega\mu_q\lambda_p}^{\cdot\,\cdot\,\cdot\,\alpha_1\nu_p} \nabla_{\alpha_1} v_{\nu_p} + \cdots \\
+ \overset{1}{V}{}_{\omega\mu_q\lambda_p}^{\cdot\,\cdot\,\cdot\,\alpha_{q-1}\nu_p} \nabla_{\alpha_{q-1}} v_{\nu_p} = 0.
\end{aligned}
\tag{10.5}
$$

Differentiation of this equation and elimination of the qth derivatives by means of (10.4) leads to an infinite set of equations of the same form

$$
\begin{aligned}
\overset{0y}{V}_{\omega_y\mu_q\lambda_p} + \overset{y}{V}{}_{\omega_y\mu_q\lambda_p}^{\cdot\,\cdot\,\cdot\,\nu_p} v_{\nu_p} + \overset{y}{V}{}_{\omega_y\mu_q\lambda_p}^{\cdot\,\cdot\,\cdot\,\alpha_1\nu_p} \nabla_{\alpha_1} v_{\nu_p} + \cdots \\
+ \overset{y}{V}{}_{\omega_y\mu_q\lambda_p}^{\cdot\,\cdot\,\cdot\,\alpha_{q-1}\nu_p} \nabla_{\alpha_{q-1}} v_{\nu_p} = 0 \quad (y = 1, 2, \cdots).
\end{aligned}
\tag{10.6}
$$

In all the preceding equations of this section quantities with a suffix 0 vanish for $\overset{0}{u}_{\mu\lambda_p} = 0$. The quantities U and V in (10.4) and (10.6) are connected by the equations

$$
\begin{aligned}
\overset{01}{V}_{\omega\mu_q\lambda_p} &= \nabla_{[\omega} \overset{0}{U}_{\mu_q]\mu_{q-1}\lambda_p} + \overset{0}{U}_{[\omega|\beta_{q-1}\gamma_p|} U_{\mu_q]\mu_{q-1}\lambda_p}^{\cdot\,\cdot\,\cdot\,\beta_{q-1}\gamma_p}, \\
\overset{1}{V}{}_{\omega\mu_q\lambda_p}^{\cdot\,\cdot\,\cdot\,\nu_p} &= \nabla_{[\omega} U_{\mu_q]\mu_{q-1}\lambda_p}^{\cdot\,\cdot\,\cdot\,\nu_p} + U_{[\omega|\beta_{q-1}\gamma_p|}^{\cdot\,\cdot\,\cdot\,\nu_p} U_{\mu_q]\mu_{q-1}\lambda_p}^{\cdot\,\cdot\,\cdot\,\beta_{q-1}\gamma_p}, \\
\overset{1}{V}{}_{\omega\mu_q\lambda_p}^{\cdot\,\cdot\,\cdot\,\alpha_z\nu_p} &= \nabla_{[\omega} U_{\mu_q]\mu_{q-1}\lambda_p}^{\cdot\,\cdot\,\cdot\,\alpha_z\nu_p} + A_{[\omega}^{\alpha_z} U_{\mu_q]\mu_{q-1}\lambda_p}^{\cdot\,\cdot\,\cdot\,\alpha_{z-1}\nu_p} \\
&\quad + U_{[\omega|\beta_{q-1}\gamma_p|}^{\cdot\,\cdot\,\cdot\,\alpha_z\nu_p} U_{\mu_q]\mu_{q-1}\lambda_p}^{\cdot\,\cdot\,\cdot\,\beta_{q-1}\gamma_p} - \tfrac{1}{2}\lambda R_{\omega\mu_q\lambda_p}^{\cdot\,\cdot\,\cdot\,\alpha_{q-1}\nu_p} \\
&\qquad (z = 1, \cdots, q-1;\ \lambda = 0,\ z < q-1;\ \lambda = 1,\ z = q-1);
\end{aligned}
\tag{10.7}
$$

$$
\begin{aligned}
&\overset{0,y+1}{V}_{\xi\omega_y\mu_q\lambda_p} &&= \nabla_\xi \overset{0y}{V}_{\omega_y\mu_q\lambda_p} &&+ \overset{0}{U}_{\xi\beta_{q-1}\gamma_p} \overset{y}{V}{}_{\omega_y\mu_q\lambda_p}^{\cdot\;\cdot\;\cdot\;\beta_{q-1}\gamma_p}, \\
&\overset{y+1}{V}{}_{\xi\omega_y\mu_q\lambda_p}^{\cdot\;\cdot\;\cdot\;\cdot\;\nu_p} &&= \nabla_\xi \overset{y}{V}{}_{\omega_y\mu_q\lambda_p}^{\cdot\;\cdot\;\cdot\;\nu_p} &&+ U_{\xi\beta_{q-1}\gamma_p}^{\cdot\;\cdot\;\;\;\cdot\;\nu_p} \overset{y}{V}{}_{\omega_y\mu_q\lambda_p}^{\cdot\;\cdot\;\cdot\;\beta_{q-1}\gamma_p}, \\
(10.8)\qquad &\overset{y+1}{V}{}_{\xi\omega_y\mu_q\lambda_p}^{\cdot\;\cdot\;\cdot\;\cdot\;\alpha_z\nu_p} &&= \nabla_\xi \overset{y}{V}{}_{\omega_y\mu_q\lambda_p}^{\cdot\;\cdot\;\cdot\;\alpha_z\nu_p} &&+ A_\xi^{\alpha_z} \overset{y}{V}{}_{\omega_y\mu_q\lambda_p}^{\cdot\;\cdot\;\cdot\;\alpha_{z-1}\nu_p} \\
& && &&+ U_{\xi\beta_{q-1}\gamma_p}^{\cdot\;\cdot\;\;\;\cdot\;\alpha_z\nu_p} \overset{y}{V}{}_{\omega_y\mu_q\lambda_p}^{\cdot\;\cdot\;\cdot\;\beta_{q-1}\gamma_p} \\
& && &&(y = 1, 2, \cdots; \quad z = 1, \cdots, q-1).
\end{aligned}
$$

Now we regard the equations (10.1), (10.2) and the first t equations (10.6) only from the algebraic point of view as equations for the determination of the unknown quantities $v_{\lambda_p}, \nabla_{\mu_1} v_{\lambda_p}, \cdots, \nabla_{\mu_{q-1}} v_{\lambda_p}$. These quantities must also satisfy the identities

$$
\begin{aligned}
&\nabla_{\mu^x[\mu_y\mu_z]\mu_{x-1}} v_{\lambda_p} = \tfrac{1}{2} \nabla_{\mu_x} R_{\mu_y\mu_z\lambda_p}^{\cdot\;\cdot\;\cdot\;\alpha_{z-1}\nu_p} \nabla_{\alpha_{z-1}} v_{\nu_p} \\
(10.9)\qquad &\qquad = \tfrac{1}{2} \nabla_{\mu_x} R_{\mu_y\lambda_p}^{\cdot\;\cdot\;\alpha_{z-1}\nu_p} \nabla_{\alpha_{z-1}} v_{\nu_p}; \\
&x = 2, \cdots, q-1;\ y = 2, \cdots, x;\ z = y-1).
\end{aligned}
$$

We assume that (10.1), (10.2), the first t equations (10.6) and (10.9) admit a solution $\overset{0}{v}_{\lambda_p}, \overset{0}{v}_{\mu_1\lambda_p}, \cdots, \overset{0}{v}_{\mu_{q-1}\lambda_p}$. Then every solution of these equations can be written in the form

$$
(10.10)\qquad v_{\mu_y\lambda_p} = \overset{0}{v}_{\mu_y\lambda_p} + \underset{1}{\alpha} \overset{1}{v}_{\mu_y\lambda_p} + \cdots + \underset{s}{\alpha} \overset{s}{v}_{\mu_y\lambda_p} \qquad (y = 1, \cdots, q-1),
$$

where $\overset{1}{v}_{\mu_y\lambda_p}, \cdots, v_{\mu_y\lambda_p}, y = 1, \cdots, q-1$ form a *fundamental set* of solutions of (10.9) and the *reduced* equations (10.1), (10.2), (10.6, first t equations), viz. the equations found by setting $\overset{0}{u}_{\mu\lambda_p} = 0$, and $\underset{1}{\alpha}, \cdots, \underset{s}{\alpha}$ are arbitrary functional parameters. Further we assume that the $(t+1)$th equation (10.6) is an algebraic consequence of (10.1), (10.2), (10.9) and the first t equations (10.6). When the system (10.1), (10.2), (10.6), (10.9) is algebraically consistent, there will certainly be a number t with these properties. Setting in the first t equations (10.6) $v_{\lambda_p} = \overset{0}{v}_{\lambda_p}$, $\nabla_{\mu_1} v_{\lambda_p} = \overset{0}{v}_{\mu_1\lambda_p}$ etc., we have by differentiation

$$
\begin{aligned}
&\nabla_\xi \overset{0y}{V}_{\omega_q\mu_q\nu_p} + \Big(\nabla_\xi \overset{y}{V}{}_{\omega_y\mu_q\lambda_p}^{\cdot\;\cdot\;\cdot\;\nu_p}\Big) \overset{0}{v}_{\nu_p} + \Big(\nabla_\xi \overset{y}{V}{}_{\omega_y\mu_q\lambda_p}^{\cdot\;\cdot\;\cdot\;\alpha_1\nu_p}\Big) \overset{0}{v}_{\alpha_1\nu_p} + \cdots \\
(10.11)\qquad &\quad + \Big(\nabla_\xi \overset{y}{V}{}_{\omega_y\mu_q\lambda_p}^{\cdot\;\cdot\;\cdot\;\alpha_{q-1}\nu_p}\Big) \overset{0}{v}_{\alpha_{q-1}\nu_p} + \overset{y}{V}{}_{\omega_y\mu_q\lambda_p}^{\cdot\;\cdot\;\cdot\;\nu_p} \nabla_\xi \overset{0}{v}_{\nu_p} + \cdots \\
&\quad + \overset{y}{V}{}_{\omega_y\mu_q\lambda_p}^{\cdot\;\cdot\;\cdot\;\alpha_{q-1}\nu_p} \nabla_\xi \overset{0}{v}_{\alpha_{q-1}\nu_p} = 0 \qquad (y = 1, \cdots, t).
\end{aligned}
$$

Now the second to the $(t+1)$th equation (10.6) can be put into the form

$$
\begin{aligned}
(10.12)\qquad & \nabla_\xi \overset{0y}{V}_{\omega_y\mu_q\lambda_p} + \overset{0}{U}_{\xi\beta_{q-1}\gamma_p} \overset{y}{V}{}_{\omega_y\mu_q\lambda_p}^{\cdot\ \cdot\ \cdot\ \beta_{q-1}\gamma_p} \\
& + \Big(\nabla_\xi \overset{y}{V}{}_{\omega_y\mu_q\lambda_p}^{\cdot\ \cdot\ \cdot\ \nu_p} + U_{\xi\beta_{q-1}\gamma_p}^{\cdot\cdot\ \ \ \cdot\ \nu_p} \overset{y}{V}{}_{\omega_y\mu_q\lambda_p}^{\cdot\ \cdot\ \cdot\ \beta_{q-1}\gamma_p}\Big) v_{\nu_p} \\
& + \Big(\nabla_\xi \overset{y}{V}{}_{\omega_y\mu_q\lambda_p}^{\cdot\ \cdot\ \cdot\ \alpha_1\nu_p} + U_{\xi\beta_{q-1}\gamma_p}^{\cdot\cdot\ \ \ \cdot\ \alpha_1\nu_p} \overset{y}{V}{}_{\omega_y\mu_q\lambda_p}^{\cdot\ \cdot\ \cdot\ \beta_{q-1}\gamma_p} \\
& \qquad\qquad + A_\xi^{\alpha_1} \overset{y}{V}{}_{\omega_y\mu_q\lambda_p}^{\cdot\ \cdot\ \cdot\ \nu_p}\Big) v_{\alpha_1\nu_p} + \cdots \\
& + \Big(\nabla_\xi \overset{y}{V}{}_{\omega_y\mu_q\lambda_p}^{\cdot\ \cdot\ \cdot\ \alpha_{q-1}\nu_p} + U_{\xi\beta_{q-1}\gamma_p}^{\cdot\cdot\ \ \ \cdot\ \alpha_{q-1}\nu_p} \overset{y}{V}{}_{\omega_y\mu_q\lambda_p}^{\cdot\ \cdot\ \cdot\ \beta_{q-1}\gamma_p} \\
& \qquad\qquad + A_\xi^{\alpha_{q-1}} \overset{y}{V}{}_{\omega_y\mu_q\lambda_p}^{\cdot\ \cdot\ \cdot\ \alpha_{q-2}\nu_p}\Big) v_{\alpha_{q-1}\nu_p} = 0, \\
& \qquad\qquad\qquad\qquad\qquad\qquad y = 1, \cdots, t,
\end{aligned}
$$

in consequence of (10.8). Setting in (10.12)

$$
\begin{aligned}
v_{\lambda_p} &= \overset{0}{v}_{\lambda_p}, \\
\nabla_{\mu_1} v_{\lambda_p} &= \overset{0}{v}_{\mu_1\lambda_p} \text{ etc.}
\end{aligned}
$$

and subtracting (10.11) from the so obtained equations we have

$$
\begin{aligned}
(10.13)\qquad & \overset{y}{V}{}_{\omega_y\mu_q\lambda_p}^{\cdot\ \cdot\ \cdot\ \nu_p}\Big(\overset{0}{v}_{\xi\nu_p} - \nabla_\xi \overset{0}{v}_{\nu_p}\Big) + \cdots + \overset{y}{V}{}_{\omega_y\mu_q\lambda_p}^{\cdot\ \cdot\ \cdot\ \alpha_{q-2}\nu_p}\Big(\overset{0}{v}_{\xi\alpha_{q-2}\nu_p} - \nabla_\xi \overset{0}{v}_{\alpha_{q-2}\nu_p}\Big) \\
& \qquad + \overset{y}{V}{}_{\omega_y\mu_q\lambda_p}^{\cdot\ \cdot\ \cdot\ \alpha_{q-1}\nu_p}\Big(\overset{0}{U}_{\xi\alpha_{q-1}\nu_p} + U_{\xi\alpha_{q-1}\nu_p}^{\cdot\cdot\ \ \ \cdot\ \gamma_p} \overset{0}{v}_{\gamma_p} + \cdots \\
& \qquad + U_{\xi\alpha_{q-1}\nu_p}^{\cdot\cdot\ \ \ \cdot\ \beta_{q-1}\gamma_p} \overset{0}{v}_{\beta_{q-1}\gamma_p} - \nabla_\xi \overset{0}{v}_{\alpha_{q-1}\nu_p}\Big) = 0.
\end{aligned}
$$

Hence the quantities

$$
\begin{aligned}
& a^\xi\Big(\overset{0}{v}_{\xi\lambda_p} - \nabla_\xi \overset{0}{v}_{\lambda_p}\Big), \\
& \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\
(10.14)\qquad & a^\xi\Big(\overset{0}{v}_{\xi\alpha_{q-2}\lambda_p} - \nabla_\xi \overset{0}{v}_{\alpha_{q-2}\lambda_p}\Big), \\
& a^\xi\Big(\overset{0}{U}_{\xi\alpha_{q-1}\lambda_p} + U_{\xi\alpha_{q-1}\lambda_p}^{\cdot\cdot\ \ \ \cdot\ \gamma_p} \overset{0}{v}_{\gamma_p} + \cdots + U_{\xi\alpha_{q-1}\lambda_p}^{\cdot\cdot\ \ \ \cdot\ \beta_{q-1}\gamma_p} \overset{0}{v}_{\beta_{q-1}\gamma_p} - \nabla_\xi \overset{0}{v}_{\alpha_{q-1}\lambda_p}\Big),
\end{aligned}
$$

where a^ξ is an arbitrary vector, form a solution of the first t *reduced* equations (10.6). We will now prove that these same quantities also satisfy (10.9) and the *reduced* equations (10.1), (10.2).

Because of (10.9) the equations (10.1, 2, 3) are equivalent to equations of the form

$$
\begin{aligned}
(10.15)\qquad P\nabla_{\mu_x} v_{\lambda_p} &= \nabla_{\mu_x} \overset{0}{u}_{\mu_1\lambda_p} + \overset{x}{W}{}_{\mu_x\lambda_p}^{\cdot\ \cdot\ \nu_p} v_{\nu_p} + \cdots \\
&\quad + \overset{x}{W}{}_{\mu_x\lambda_p}^{\cdot\ \cdot\ \alpha_{x-1}\nu_p} \nabla_{\alpha_{x-1}} v_{\nu_p} = 0 \quad (x = 1, \cdots, q),
\end{aligned}
$$

where

$$
\begin{aligned}
&\overset{x+1}{W}{}^{\cdot\,\cdot\,\cdot\,\nu_p}_{\xi\mu_x\lambda_p} = \nabla_\xi \overset{x}{W}{}^{\cdot\,\cdot\,\nu_p}_{\mu_x\lambda_p},\\
(10.16)\qquad &\overset{x+1}{W}{}^{\cdot\,\cdot\,\alpha_y\nu_p}_{\xi\mu_x\lambda_p} = \nabla_\xi \overset{x}{W}{}^{\cdot\,\cdot\,\alpha_y\nu_p}_{\mu_x\lambda_p} + A^{\alpha_y}_\xi \overset{x}{W}{}^{\cdot\,\cdot\,\alpha_{y-1}\nu_p}_{\mu_x\lambda_p},\\
&\overset{x+1}{W}{}^{\cdot\,\cdot\,\alpha_x\nu_p}_{\xi\mu_x\lambda_p} = A^{\alpha_x}_\xi \overset{x}{W}{}^{\cdot\,\cdot\,\alpha_{x-1}\nu_p}_{\mu_x\lambda_p}
\end{aligned}
$$

$$(x = 1,\cdots,q-1;\quad y = 1,\cdots,x-1).$$

Substituting $\overset{0}{v}_{\lambda_p}$, $\overset{0}{v}_{\mu_1\lambda_p}$ etc. in the first $q-1$ equations (10.15) we have by differentiation of the first $q-2$ equations

$$
\begin{aligned}
P\nabla_\xi \overset{0}{v}_{\mu_x\lambda_p} &= \nabla_{\xi\mu_x}\overset{0}{u}_{\mu_1\lambda_p} + \left(\nabla_\xi \overset{x}{W}{}^{\cdot\,\cdot\,\nu_p}_{\mu_x\lambda_p}\right)\overset{0}{v}_{\nu_p} + \overset{x}{W}{}^{\cdot\,\cdot\,\nu_p}_{\mu_x\lambda_p}\nabla_\xi \overset{0}{v}_{\nu_p} + \cdots\\
(10.17)\qquad &+ \left(\nabla_\xi \overset{x}{W}{}^{\cdot\,\cdot\,\alpha_{x-1}\nu_p}_{\mu_x\lambda_p}\right)\overset{0}{v}_{\alpha_{x-1}\nu_p} + \overset{x}{W}{}^{\cdot\,\cdot\,\alpha_{x-1}\nu_p}_{\mu_x\lambda_p}\nabla_\xi \overset{0}{v}_{\alpha_{x-1}\nu_p}
\end{aligned}
$$

$$(x = 1,\cdots,q-2).$$

Subtracting these equations from the second to the $(q-1)$th equation (inclusive) obtained from (10.15) by substituting $\overset{0}{v}_{\lambda_p}$, $\overset{0}{v}_{\mu_1\lambda_p}$ etc. we have in consequence of (10.16)

$$
\begin{aligned}
P(\overset{0}{v}_{\xi\mu_x\lambda_p} - \nabla_\xi \overset{0}{v}_{\mu_x\lambda_p}) &= \overset{x}{W}{}^{\cdot\,\cdot\,\nu_p}_{\mu_x\lambda_p}(\overset{0}{v}_{\xi\nu_p} - \nabla_\xi \overset{0}{v}_{\nu_p}) + \cdots\\
(10.18)\qquad &+ \overset{x}{W}{}^{\cdot\,\cdot\,\alpha_{x-1}\nu_p}_{\mu_x\lambda_p}(\overset{0}{v}_{\xi\alpha_{x-1}\nu_p} - \nabla_\xi \overset{0}{v}_{\alpha_{x-1}\nu_p})\quad (x = 1,\cdots,q-2).
\end{aligned}
$$

Differentiating the $(q-1)$th equation (10.15) after having substituted $\overset{0}{v}_{\lambda_p}$ etc. in it, we have

$$
\begin{aligned}
P\nabla_\xi \overset{0}{v}_{\mu_{q-1}\lambda_p} &= \nabla_{\xi\mu_{q-1}}\overset{0}{u}_{\mu_1\lambda_p} + \left(\nabla_\xi \overset{q-1}{W}{}^{\cdot\,\cdot\,\nu_p}_{\mu_{q-1}\lambda_p}\right)\overset{0}{v}_{\nu_p} + \overset{q-1}{W}{}^{\cdot\,\cdot\,\nu_p}_{\mu_{q-1}\lambda_p}\nabla_\xi \overset{0}{v}_{\nu_p} + \cdots\\
(10.19)\qquad &+ \left(\nabla_\xi \overset{q-1}{W}{}^{\cdot\,\cdot\,\alpha_{q-2}\nu_p}_{\mu_{q-1}\lambda_p}\right)\overset{0}{v}_{\alpha_{q-2}\nu_p} + \overset{q-1}{W}{}^{\cdot\,\cdot\,\alpha_{q-2}\nu_p}_{\mu_{q-1}\lambda_p}\nabla_\xi \overset{0}{v}_{\alpha_{q-2}\nu_p},
\end{aligned}
$$

or, in consequence of (10.16),

$$
\begin{aligned}
&P\nabla_\xi \overset{0}{v}_{\mu_{q-1}\lambda_p} - \nabla_{\xi\mu_{q-1}}\overset{0}{u}_{\mu_1\lambda_p} - \overset{q}{W}{}^{\cdot\,\cdot\,\cdot\,\nu_p}_{\xi\mu_{q-1}\lambda_p}\overset{0}{v}_{\nu_p}\\
&\qquad - \overset{q}{W}{}^{\cdot\,\cdot\,\cdot\,\alpha_1\nu_p}_{\xi\mu_{q-1}\lambda_p}\overset{0}{v}_{\alpha_1\nu_p} + \overset{q-1}{W}{}^{\cdot\,\cdot\,\nu_p}_{\mu_{q-1}\lambda_p}\overset{0}{v}_{\xi\nu_p} - \cdots\\
(10.20)\qquad &\qquad - \overset{q}{W}{}^{\cdot\,\cdot\,\cdot\,\alpha_{q-1}\nu_p}_{\xi\mu_{q-1}\lambda_p}\overset{0}{v}_{\alpha_{q-1}\nu_p} + \overset{q-1}{W}{}^{\cdot\,\cdot\,\alpha_{q-2}\nu_p}_{\mu_{q-1}\lambda_p}\overset{0}{v}_{\xi\alpha_{q-2}\nu_p}\\
&= \overset{q-1}{W}{}^{\cdot\,\cdot\,\nu_p}_{\mu_{q-1}\lambda_p}\nabla_\xi \overset{0}{v}_{\nu_p} + \cdots + \overset{q-1}{W}{}^{\cdot\,\cdot\,\alpha_{q-2}\nu_p}_{\mu_{q-1}\lambda_p}\nabla_\xi \overset{0}{v}_{\alpha_{q-2}\nu_p}.
\end{aligned}
$$

Now the qth equation (10.15) is equivalent to (10.4). Hence

$$
\begin{aligned}
&\nabla_{\xi\mu_{q-1}} \overset{0}{u}_{\mu_1\lambda_p} + \overset{q}{W}{}_{\xi\mu_{q-1}\lambda_p}^{\cdot\cdot\;\cdot\;\nu_p} \overset{0}{v}_{\nu_p} + \cdots + \overset{q}{W}{}_{\xi\mu_{q-1}\lambda_p}^{\cdot\cdot\;\cdot\;\alpha_{q-1}\nu_p} \overset{0}{v}_{\alpha_{q-1}\nu_p}\\
&\quad = P\Big(\overset{0}{U}_{\xi\mu_{q-1}\lambda_p} + U_{\xi\mu_{q-1}\lambda_p}^{\cdot\cdot\;\cdot\;\nu_p} \overset{0}{v}_{\nu_p} + \cdots + U_{\xi\mu_{q-1}\lambda_p}^{\cdot\cdot\;\cdot\;\alpha_{q-1}\nu_p} \overset{0}{v}_{\alpha_{q-1}\nu_p}\Big).
\end{aligned}
\tag{10.21}
$$

Hence (10.19) is equivalent to

$$
\begin{aligned}
&P\Big(\overset{0}{U}_{\xi\mu_{q-1}\lambda_p} + U_{\xi\mu_{q-1}\lambda_p}^{\cdot\cdot\;\cdot\;\nu_p} \overset{0}{v}_{\nu_p} + \cdots + U_{\xi\mu_{q-1}\lambda_p}^{\cdot\cdot\;\cdot\;\alpha_{q-1}\nu_p} \overset{0}{v}_{\alpha_{q-1}\nu_p} - \nabla_\xi \overset{0}{v}_{\mu_{q-1}\lambda_p}\Big)\\
&\quad = \overset{q-1}{W}{}_{\mu_{q-1}\lambda_p}^{\cdot\;\cdot\;\nu_p}(\overset{0}{v}_{\xi\nu_p} - \nabla_\xi \overset{0}{v}_{\nu_p}) + \cdots + \overset{q-1}{W}{}_{\mu_{q-1}\lambda_p}^{\cdot\;\cdot\;\alpha_{q-2}\nu_p}(\overset{0}{v}_{\xi\alpha_{q-2}\nu_p} - \nabla_\xi \overset{0}{v}_{\alpha_{q-2}\nu_p}).
\end{aligned}
\tag{10.22}
$$

From (10.18) and (10.22) it follows that the quantities (10.14) satisfy the *reduced* equations (10.1, 2), q. e. d.

Substituting $\overset{0}{v}_{\lambda_p}$ etc. in (10.9) we obtain

$$
\begin{aligned}
&\overset{0}{v}_{\mu_x[\mu_y\mu_z]\mu_{z-1}\lambda_p} = \tfrac{1}{2}\nabla'_{\mu_x} R_{\mu_y\lambda_p}^{\cdot\;\cdot\;\alpha_{z-1}\nu_p} \overset{0}{v}_{\alpha_{z-1}\nu_p}\\
&\qquad (x = 2, \cdots, q-1; \quad y = 2, \cdots, x; \quad z = y-1),
\end{aligned}
\tag{10.23}
$$

where on the right side ∇' is a real differential operator equivalent to ∇ for R but a symbolic one for $\overset{0}{v}$, $\nabla'_{\mu_i} \overset{0}{v}_{\alpha_{z-1}\lambda_p}$ being a symbolic notation for $\overset{0}{v}_{\mu_i\alpha_{z-1}\lambda_p}$, $i = 0, \cdots, x$. Differentiating the first $q-3$ equations (10.23) for some fixed values of y we have

$$
\begin{aligned}
\nabla_\xi \overset{0}{v}_{\mu_x[\mu_y\mu_z]\mu_{z-1}\lambda_p} &= \tfrac{1}{2}\overset{1}{\nabla}_\xi \nabla'_{\mu_x} R_{\mu_y\lambda_p}^{\cdot\;\cdot\;\alpha_{z-1}\nu_p} \overset{0}{v}_{\alpha_{z-1}\nu_p}\\
&\quad + \tfrac{1}{2}\overset{2}{\nabla}_\xi \nabla'_{\mu_x} R_{\mu_y\lambda_p}^{\cdot\;\cdot\;\alpha_{z-1}\nu_p} \overset{0}{v}_{\alpha_{z-1}\nu_p}\\
&\qquad (x = 2, \cdots, q-2; \; y = 2, \cdots, x; \; z = y-1)
\end{aligned}
\tag{10.24}
$$

where $\overset{1}{\nabla}$ differentiates only $R_{\mu_y\lambda_p}^{\cdot\;\cdot\;\alpha_{z-1}\nu_p}$ and $\overset{2}{\nabla}$ only $\overset{0}{v}_{\mu_x\alpha_{z-1}\lambda_p}$. $\overset{2}{\nabla}_\xi \overset{0}{v}_{\mu_i\alpha_{z-1}\nu_p}$ is *not* to be replaced by $\overset{0}{v}_{\xi\mu_i\alpha_{z-1}\nu_p}$, $\overset{2}{\nabla}$ being a real differentiation, not a symbolic one. Subtracting these equations from the second to the $(q-1)$th equation (10.23) (inclusive) for the same value of y we obtain

$$
\begin{aligned}
&\overset{0}{v}_{\xi\mu_x[\mu_y\mu_z]\mu_{z-1}\lambda_p} - \nabla_\xi \overset{0}{v}_{\mu_x[\mu_y\mu_z]\mu_{z-1}\lambda_p}\\
&\qquad = \tfrac{1}{2}\nabla'_\xi \nabla'_{\mu_x} R_{\mu_y\lambda_p}^{\cdot\;\cdot\;\alpha_{z-1}\nu_p} \overset{0}{v}_{\alpha_{z-1}\nu_p}\\
&\quad - \tfrac{1}{2}\overset{1}{\nabla}_\xi \nabla'_{\mu_x} R_{\mu_y\lambda_p}^{\cdot\;\cdot\;\alpha_{z-1}\nu_p} \overset{0}{v}_{\alpha_{z-1}\nu_p}\\
&\quad - \tfrac{1}{2}\overset{2}{\nabla}_\xi \nabla'_{\mu_x} R_{\mu_y\lambda_p}^{\cdot\;\cdot\;\alpha_{z-1}\nu_p} \overset{0}{v}_{\alpha_{z-1}\nu_p}\\
&\qquad = \tfrac{1}{2}(\overset{2}{\nabla}{}'_\xi - \overset{2}{\nabla}_\xi) \nabla'_{\mu_x} R_{\mu_y\lambda_p}^{\cdot\;\cdot\;\alpha_{z-1}\nu_p} \overset{0}{v}_{\alpha_{z-1}\nu_p}.
\end{aligned}
\tag{10.25}
$$

Differentiating the $(q-2)$th equation (10.23) for some fixed value of y we have

$$(10.26)\qquad \nabla_\xi \overset{0}{v}_{\mu_{q-1}[\mu_y\mu_z]\mu_{z-1}\lambda_p} = \tfrac{1}{2}\nabla_\xi \nabla'_{\mu_{q-1}} R_{\mu_y\lambda_p}^{\cdot\ \cdot\ \alpha_{z-1}\nu_p}\, \overset{0}{v}_{\alpha_{z-1}\nu_p}.$$

Now from (10.4) we deduce

$$\begin{aligned}
&\nabla_{\xi\mu_{q-1}[\mu_y\mu_z]\mu_{z-1}}\, v_{\lambda_p} = \tfrac{1}{2}\nabla_{\xi\mu_{q-1}} R_{\mu_y\lambda_p}^{\cdot\ \cdot\ \alpha_{z-1}\nu_p}\, \nabla_{\alpha_{z-1}} v_{\nu_p}\\
(10.27)\qquad &= \overset{0}{U}_{\xi\mu_{q-1}[\mu_y\mu_z]\mu_{z-1}\lambda_p} + U_{\xi\mu_{q-1}[\mu_y\mu_z]\mu_{z-1}\lambda_p}^{\cdot\cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \nu_p}\, v_{\nu_p} + \cdots\\
&\qquad + U_{\xi\mu_{q-1}[\mu_y\mu_z]\mu_{z-1}\lambda_p}^{\cdot\cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \alpha_{q-1}\nu_p}\, \nabla_{\alpha_{q-1}} v_{\nu_p},
\end{aligned}$$

hence it follows that

$$(10.28)\qquad \begin{aligned}
&\overset{0}{U}_{\xi\mu_{q-1}[\mu_y\mu_z]\mu_{z-1}\lambda_p} = 0,\\
&U_{\xi\mu_{q-1}[\mu_y\mu_z]\mu_{z-1}\lambda_p}^{\cdot\cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \alpha_i\nu_p} = 0 \qquad (i = 1, \cdots, z-2,\ q-1),
\end{aligned}$$

and that there exist such relations between the quantities U and R in (10.27) that for every set of quantities $\overset{0}{v}_{\nu_p}$, etc. which satisfies (10.9) the following equation holds:

$$\begin{aligned}
&\tfrac{1}{2}\nabla'_{\xi\mu_{q-1}} R_{\mu_y\lambda_p}^{\cdot\ \cdot\ \alpha_{z-1}\nu_p}\, \overset{0}{v}_{\alpha_{z-1}\nu_p}\\
(10.29)\qquad &= \overset{0}{U}_{\xi\mu_{q-1}[\mu_y\mu_z]\mu_{z-1}\lambda_p} + U_{\xi\mu_{q-1}[\mu_y\mu_z]\mu_{z-1}\lambda_p}^{\cdot\cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \nu_p}\, \overset{0}{v}_{\nu_p} + \cdots\\
&\qquad + U_{\xi\mu_{q-1}[\mu_y\mu_z]\mu_{z-1}\lambda_p}^{\cdot\cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \alpha_{q-1}\nu_p}\, \overset{0}{v}_{\alpha_{q-1}\nu_p}.
\end{aligned}$$

From (10.26) and (10.29) we get

$$\begin{aligned}
&\overset{0}{U}_{\xi\mu_{q-1}[\mu_y\mu_z]\mu_{z-1}\lambda_p} + U_{\xi\mu_{q-1}[\mu_y\mu_z]\mu_{z-1}\lambda_p}^{\cdot\cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \nu_p}\, \overset{0}{v}_{\nu_p} + \cdots\\
(10.30)\qquad &+ U_{\xi\mu_{q-1}[\mu_y\mu_z]\mu_{z-1}\lambda_p}^{\cdot\cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \alpha_{q-1}\nu_p}\, \overset{0}{v}_{\alpha_{q-1}\nu_p} - \nabla_\xi\, \overset{0}{v}_{\mu_{q-1}[\mu_y\mu_z]\mu_{z-1}\lambda_p}\\
&= \tfrac{1}{2}(\overset{2}{\nabla}{}'_\xi - \overset{2}{\nabla}_\xi)\, \nabla'_{\mu_{q-1}} R_{\mu_y\lambda_p}^{\cdot\ \cdot\ \alpha_{z-1}\nu_p}\, \overset{0}{v}_{\alpha_{z-1}\nu_p}.
\end{aligned}$$

From (10.25), (10.27) and (10.30) follows that the quantities (10.14) satisfy also the equations (10.9), q. e. d. Hence these quantities satisfy the equations (10.9) and the reduced equations (10.1, 2) and (10.6, first t equations). The same can be proved in like manner for the quantities

$$a^{\xi}(\overset{a}{v}_{\xi\lambda_p} \quad - \nabla_{\xi}\overset{a}{v}_{\lambda_p}),$$
$$\vdots$$

(10.31)
$$a^{\xi}(\overset{a}{v}_{\xi\alpha_{q-2}\lambda_p} - \nabla_{\xi}\overset{a}{v}_{\alpha_{q-2}\lambda_p}),$$
$$a^{\xi}\left(U_{\xi\alpha_{q-1}\lambda_p}^{\cdot\cdot\quad\cdot\gamma_p}\overset{a}{v}_{\gamma_p} + \cdots + U_{\xi\alpha_{q-1}\lambda_p}^{\cdot\cdot\quad\cdot\beta_{q-1}\gamma_p}\overset{a}{v}_{\beta_{q-1}\gamma_p} - \nabla_{\xi}\overset{a}{v}_{\alpha_{q-1}\lambda_p}\right)$$
$$(a = 1, \cdots, s).$$

A fundamental set of solutions of the reduced equations (10.1, 2) and (10.6, first t equations) being $\overset{1}{v}_{\lambda_p}, \cdots, \overset{s}{v}_{\lambda_p}$, etc., there must exist equations of the form

(10.32)
$$\overset{i}{v}_{\xi\nu_p} \quad - \nabla_{\xi}\overset{i}{v}_{\nu_p} \quad = \sum_b^{1,\cdots,s}\overset{ib}{p}_{\xi}\overset{b}{v}_{\nu_p},$$
$$\overset{i}{v}_{\xi\alpha_{q-2}\nu_p} - \nabla_{\xi}\overset{i}{v}_{\alpha_{q-2}\nu_p} = \sum_b^{1,\cdots,s}\overset{ib}{p}_{\xi}\overset{b}{v}_{\alpha_{q-2}\nu_p},$$
$$\overset{i}{\varepsilon}\overset{0}{U}_{\xi\alpha_{q-1}\nu_p} + U_{\xi\alpha_{q-1}\nu_p}^{\cdot\cdot\quad\cdot\gamma_p}\overset{i}{v}_{\gamma_p} + \cdots + U_{\xi\alpha_{q-1}\nu_p}^{\cdot\cdot\quad\cdot\beta_{q-1}\gamma_p}\overset{i}{v}_{\beta_{q-1}\gamma_p} - \nabla_{\xi}\overset{i}{v}_{\alpha_{q-1}\nu_p}$$
$$= \sum_b^{1,\cdots,s}\overset{ib}{p}_{\xi}\overset{b}{v}_{\alpha_{q-1}\nu_p}$$
$$\left(i = 0, 1, \cdots, s; \quad \overset{i}{\varepsilon} = \begin{cases} 1, & i = 0 \\ 0, & i \neq 0 \end{cases}\right).$$

By differentiation and alternation we get from (10.32)

(10.33)
$$\nabla_{[\eta}\overset{i}{v}_{\xi]\alpha_x\nu_p} - \nabla_{[\eta}\nabla_{\xi]}\overset{i}{v}_{\alpha_x\nu_p} = \sum_b^{1,\cdots,s}(\nabla_{[\eta}\overset{ib}{p}_{\xi]})\overset{b}{v}_{\alpha_x\nu_p} + \cdots$$
$$+ \sum_b^{1,\cdots,s}\overset{ib}{p}_{[\xi}\nabla_{\eta]}\overset{b}{v}_{\alpha_x\nu_p} \quad (x = 0, \cdots, q-2);$$
$$\overset{i}{\varepsilon}\nabla_{[\eta}\overset{0}{U}_{\xi]\alpha_{q-1}\nu_p} + \left(\nabla_{[\eta}U_{\xi]\alpha_{q-1}\nu_p}^{\cdot\cdot\quad\cdot\gamma_p}\right)\overset{i}{v}_{\gamma_p} + \cdots + \left(\nabla_{[\eta}U_{\xi]\alpha_{q-1}\nu_p}^{\cdot\cdot\quad\cdot\beta_{q-1}\gamma_p}\right)\overset{i}{v}_{\beta_{q-1}\gamma_p}$$
$$+ (\nabla_{[\eta}\overset{i}{v}_{|\gamma_p|})U_{\xi]\alpha_{q-1}\nu_p}^{\cdot\cdot\quad\cdot\gamma_p} + \cdots + (\nabla_{[\eta}\overset{i}{v}_{|\beta_{q-1}\gamma_p|})U_{\xi]\alpha_{q-1}\nu_p}^{\cdot\cdot\quad\cdot\beta_{q-1}\gamma_p} - \nabla_{[\eta}\nabla_{\xi]}\overset{i}{v}_{\alpha_{q-1}\nu_p}$$
$$= \sum_b^{1,\cdots,s}(\nabla_{[\eta}\overset{ib}{p}_{\xi]})\overset{b}{v}_{\alpha_{q-1}\nu_p} + \sum_b^{1,\cdots,s}\overset{ib}{p}_{[\xi}\nabla_{\eta]}\overset{b}{v}_{\alpha_{q-1}\nu_p}.$$

Because of (10.32), (10.7) and (10.5), we get from (10.33)

(10.34)
$$\sum_b^{1,\cdots,s}\nabla_{[\eta}\overset{ib}{p}_{\xi]}\overset{b}{v}_{\alpha_x\nu_p} + \sum_{b,c}^{1,\cdots,s}\overset{ic}{p}_{[\eta}\overset{cb}{p}_{\xi]}\overset{b}{v}_{\alpha_x\nu_p} \quad (x = 0, \cdots, q-1).$$

Hence

$$\nabla_{[\omega} \overset{ia}{p}_{\mu]} + \sum_{b}^{1,\dots,s} \overset{ib}{p}_{[\omega} \overset{ba}{p}_{\mu]} = 0 \qquad (i = 0, 1, \cdots, s). \tag{10.35}$$

Now we will prove that the general solution of (10.1) has the form

$$v_{\lambda_p} = \overset{0}{v}_{\lambda_p} + \underset{1}{\alpha} \overset{1}{v}_{\lambda_p} + \cdots + \underset{s}{\alpha} \overset{s}{v}_{\lambda_p}, \tag{10.36}$$

and that the parameters α can be found by the integration of a *completely integrable* system. In order that (10.1) be satisfied by (10.36) it is necessary and sufficient that

$$\nabla_\mu \overset{0}{v}_{\lambda_p} + \sum_{a}^{1,\dots,s} (\nabla_\mu \underset{a}{\alpha}) \overset{a}{v}_{\lambda_p} + \sum_{a}^{1,\dots,s} \underset{a}{\alpha} \nabla_\mu \overset{a}{v}_{\lambda_p} = \overset{0}{v}_{\mu\lambda_p} + \sum_{a}^{1,\dots,s} \underset{a}{\alpha} \overset{a}{v}_{\mu\lambda_p}, \tag{10.37}$$

or, because of (10.32),

$$-\sum \overset{0a}{p}_\mu \overset{a}{v}_{\lambda_p} + \sum_{a}^{1,\dots,s} (\nabla_\mu \underset{a}{\alpha}) \overset{a}{v}_{\lambda_p} - \sum_{ab}^{1,\dots,s} \underset{a}{\alpha} \overset{ab}{p}_\mu \overset{b}{v}_{\lambda_p} = 0, \tag{10.38}$$

or

$$\nabla_\lambda \underset{a}{\alpha} = \overset{0a}{p}_\lambda + \sum_{b}^{1,\dots,s} \underset{b}{\alpha} \overset{ba}{p}_\lambda. \tag{10.39}$$

These equations having the same form as (3.13) are completely integrable.

Hence *the integration of* (10.1) *is reduced to the algebraic solution of the system* (10.1, 2, 6, 9) *and the integration of the completely integrable system* (10.38).

DELFT, HOLLAND.

COMPLETE GROUPS OF POINTS ON A PLANE CUBIC CURVE OF GENUS ONE*

BY

M. I. LOGSDON

1. Of the great number of papers which have been published which deal with rational solutions of cubic equations in two non-homogeneous or three homogeneous variables, the majority have been concerned with special numerical examples. (See Dickson's *History of the Theory of Numbers,* vol. 2, 1920, Chap. 21, for an account of these papers.) Some few, however, have dealt with the general problem of classifying cubics with rational coefficients with reference to rational solutions. Of these, the following have been consulted in the preparation of this paper:

B. Levi, *Atti*, IV *Congresso Internazionale Matematico,* Roma, vol. 2, 1909, pp. 173–7.

B. Levi, Atti, Reale Academia delle Scienze, Torino, vol. 41 (1906), pp. 739–64; vol. 43 (1908), pp. 99–120; 413–434; 672–681.

A. Hurwitz, Vierteljahrschrift der Naturforschenden Gesellschaft, Zürich, vol. 62 (1917), pp. 207–229.

H. Poincaré, Journal de Mathématiques, ser. 5, vol. 7 (1901), pp. 161–204.

L. J. Mordell, Proceedings of the Cambridge Philosophical Society, vol. 21 (1922), pp. 181–6.

L. J. Mordell, Science Progress, London, July, 1923.

If there is a known rational point on the cubic curve, the graph of

$$f(x, y, z) = 0, \tag{1}$$

where $f(x, y, z)$ is homogeneous of degree three in the variables, other rational points may be found from this one by drawing tangents and secants. Hurwitz, in the paper quoted, page 207, calls this the *fundamental construction* and the group of points consisting of all that may be thus obtained from a given *basis* he calls a *complete group*. (A definition of basis as used by Levi will be given in § 3.)

The *anharmonic ratio of a cubic of genus one* is by definition the anharmonic ratio of the pencil of four tangents to the curve from any point on it. This ratio is independent of the point from which the tangents are drawn.†

In this paper a study is made of the geometrical configurations of rational points obtained by the fundamental construction from one or more

* Presented (in part) to the Society, December 29, 1922.

† Salmon, *Higher Plane Curves*, 3d edition, 1879, p. 144.

known rational points on cubics of genus one with *rational anharmonic ratio,* with a very brief summary of known results in the case of genus zero.

2. If the genus of the cubic is zero, in fact for the general equation $f(x, y, z) = 0$, of degree n with rational coefficients and genus zero, the problem of classification and solutions has been completely solved. By a succession of birational transformations a set of equations can be obtained,

$$\begin{aligned} f = f_0 &= 0, \\ f_1 &= 0, \\ f_2 &= 0, \\ \cdots&\cdots, \end{aligned}$$

each of degree two less than the one which precedes it in the set, but equivalent to it in the sense that to every rational solution of $f_i = 0$ $(i = 1, 2, \cdots)$ will correspond a uniquely determinable rational solution of f_{i-1} and hence of the original equation, while conversely every solution (rational) of the given equation is obtainable from those of $f_1 = 0$, $f_2 = 0$, etc. with the possible exception of the $(n-1)(n-2)/2$ double points. These latter are the fundamental points of the transformations and can always be found by solving a finite number of algebraic equations. If n is odd the final equation of the set is linear and consequently has an infinite number of rational solutions, while if n is even, say $n = 2k$, after $k-1$ steps a quadratic equation is obtained. This will have no rational solutions or an infinite number. The test is (Legendre) as follows:

If in the conic $ax^2 + by^2 + cz^2 = 0$ the a, b, c are relatively prime in pairs, a situation always obtainable by the transformation

$$\begin{aligned} x' &= mx, \\ y' &= ny, \\ z' &= pz, \end{aligned}$$

there will be one and hence an infinity of solutions if and only if a, b, c, are not all of the same sign and $-bc$, $-ca$, $-ab$ are quadratic residues of a, b, c respectively. If there are no rational solutions to the quadratic equation the original $f = 0$ has no rational solutions other than *possibly* the douple or multiple points finite in number and obtainable as simultaneous solutions of $f_x = 0$, $f_y = 0$, $f_z = 0$, the three left members designating the three partial derivatives of f. In either case the inverses of the transformations used enable us to express all the rational solutions of $f = 0$ if infinite in number rationally in terms of a rational parameter.*

* Cf. Hilbert and Hurwitz, Acta Mathematica, vol. 14 (1890–91), pp. 217–24.

3. If the genus is unity, the coördinates of points on the cubic, C, defined by (1) where $f(x, y, z)$ is a cubic ternary form with rational coefficients are expressible in terms of an elliptic parameter. The *tangential** of a known rational point, P_0, will be the third point of intersection with the cubic of the tangent

$$(2) \qquad xf_{x_0} + yf_{y_0} + zf_{z_0} = 0$$

at P_0. If two rational points are known and neither is the tangential of the other, the secant line joining the two will cut the cubic in a third rational point. If after a finite number of operations, no new points can be obtained by this construction, we say we have a complete group of rational points. There will be a finite number of points in a complete group.

It will be shown that a certain choice of coördinate representation will assure the presence of an inflexion point among the points of a complete group. If besides the inflexion point there are r points which with the inflexion have the property that all and only the points of the group may be obtained from them by the fundamental construction, the r points are called the *basis* of the group. This definition of basis is due to Levi.† It differs from that of Poincaré in that by excluding an inflexion from the r points rank becomes invariant. Thus a cubic with only one rational point is of *rank* zero (see next sentence), since the cubic is birationally equivalent to a cubic with a rational inflexion and no other rational point.

If a complete group exhausts the rational points of the cubic, its basis, r, is called the *rank* of the cubic. If there should be more than one complete group on a cubic having the respective bases $r_1, r_2, \cdots$, and no rational points not in these groups, the rank of the cubic is defined as $r_1 + r_2 + \cdots = r$. Whether there can be more than one complete group on a cubic is a question which has not been answered. That rank is finite was proved by Mordell in the 1914 paper cited. The notion of rank may be extended to a cubic with an infinite number of rational points. If there are r points which have the property that from them and the rational inflexions in complete groups, if any are present, may be obtained by the fundamental construction all of the rational points of the cubic, r is called again the rank of the cubic.

4. If only a finite number of points may be obtained from a known rational point, A_0, by the fundamental construction, the same is true if the process of finding tangentials only is used. Let $A_0, A_1, A_2, \cdots, A_{k-1}$ be rational points on C, let A_i be the tangential of A_{i-1} $(i = 1, 2, \cdots, k)$, and let α_i be the elliptic argument of the point A_i,

* J. J. Sylvester, American Journal of Mathematics, vol. 3 (1880), pp. 61–66.

† Loc. cit. vol. 41 (1906), p. 758.

(3) $$\alpha_i = \int_0^{\mu_i} \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}};$$

then, since the sum of the arguments of three collinear points on C is divisible by a period we have

$$\alpha_i \equiv -2\alpha_{i-1} \qquad \text{(modulo a period)}.$$

The set of A's may terminate in either of two ways. A point A_k may be reached ($k = 0, 1, 2, \cdots, k$), which is an inflexion point or the point A_k may coincide with one of the points $A_i (0 \leqq i \leqq k-3)$ previously obtained. The geometric configurations consisting of these $k+1$ points are called by Levi (loc. cit., vol. 43 (1898), p. 101) *arborescent* and *polygonal* respectively. In the latter case there will be a closed polygon of $k-i$ vertices. In either case the argument, α, of the general point of the set is commensurable with a period of the Jacobi functions of α which will be used in representing the coördinates of the point A after equation (1) has been transformed to a normal form. (See § 10.)

For

$$\begin{aligned}
\alpha_1 &\equiv -2\alpha_0 && \text{(modulo a period)},\\
\alpha_2 &\equiv -2\alpha_1 \equiv (-2)^2\alpha_0 && \text{(modulo a period)},\\
\alpha_3 &\equiv -2\alpha_2 \equiv (-2)^3\alpha_0 && \text{(modulo a period)},\\
&\cdots\cdots\cdots\cdots\\
\alpha_{k-1} &\equiv -2\alpha_{k-2} \equiv (-2)^{k-1}\alpha_0 && \text{(modulo a period)},
\end{aligned}$$

and finally we have the respective cases

I. $$\alpha_k \equiv (-2)^k\alpha_0 = \omega_1/3 \text{ or } \equiv 0 \qquad \text{(modulo a period)},$$

depending on whether this particular point of inflexion corresponds to zero or not (the value of the elliptic parameter corresponding to any point of inflexion in any coördinate system will be zero or one third of a period), whence

$$\alpha_0 = \frac{\omega_1}{3\cdot 2^k} \text{ or } \omega_2/(-2)_k;$$

II. $$\alpha_k \equiv (-2)^k\alpha_0 \equiv (-2)^i\alpha_0,$$

whence

$$\alpha_0 = \frac{\omega_3}{(-2^k)-(-2)^i},$$

where $\omega_1, \omega_2, \omega_3$ are notations used to designate a convenient period not divisible by two, otherwise the set would have terminated at most with A_{k-1}.

The last equation may be simplified by factoring the denominator. One factor will be $(-2)^i$, another $(-2-1)$, and the final factor, which we shall denote by M, is

$$[(-2)^{k-i-1}+(-2)^{k-i-2}+\cdots+(-2)+1], \tag{4}$$

a quantity not divisible by 2. If $k=i+3$, M will be 3. We have then, in the polygonal case, that the argument of one of the points, say A_0, in the configuration is given by

$$\alpha_0 = \frac{-\omega_3}{(-2)^i\cdot 3\cdot M} = \frac{(-1)^{i+1}\,\omega_3}{2^i\cdot 3\cdot M}. \tag{4a}$$

Thus if in the construction or computation of tangentials we are halted by reaching an inflexion, the elliptic argument of any point, A, of the group may contain in the denominator *one* factor 3 or a power of 2, but no further factor, while, second, if the set ends by the closing of a polygon with $k-i \geqq 3$ sides there will also be in the denominator another factor, M, defined in (4) which, as is evident, is different from 3 unless $k-i=3$, when the denominator is $2^i\,3^2$. Moreover there will be i points of the group which are not vertices of the closed polygon.

5. For the algebraic investigation of configurations of rational points on C, *we assume a known rational point,* which we call A_0. If A_0 is not an inflexion point, i. e. does not also satisfy the Hessian of $f(x, y, z)=0$, we can by a preliminary transformation transform C into an equivalent (in the sense of § 2) cubic with a rational inflexion. Find A_1, the tangential of A_0. If A_1 is not an inflexion call its tangential A_2. If A_2 is not an inflexion and its tangential is not an inflexion we discuss separately the two cases:

I. The tangential of A_2 is A_0,

II. The tangential of A_2 is distinct from A_0.

Take the three points A_0, A_1, A_2 as vertices of the new triangle of reference. The transformation may be written

$$T: \qquad x':y':z'::A_0A_1:A_1A_2:A_2A_0,$$

where by A_0A_1, for example, is meant the linear function whose graph is the line through the two points A_0 and A_1. In the first case the equation of the cubic becomes

$$ay'^2x'+bx'^2z'+cz'^2y'+dx'y'z'=0.$$

Now transform to new variables x, y, z by the Cremona transformation

$$x : y : z :: x'y' : z'^2 : z'x',$$

and get, apart from the factor yz^2 which does not count in the transformation,

$$ax^2y + bz^3 + cxy^2 + dxyz = 0, \tag{5}$$

which has inflexion points at $(0, 1, 0)$, $(1, 0, 0)$, and $(c, -a, 0)$. In the second case transformation T reduces the cubic to

$$ax'^2y' + bx'^2z' + cx'y'^2 + dy'z'^2 + ex'y'z' = 0,$$

which by the Cremona transformation

$$x : y : z :: x'^2 : x'y' : y'z' \tag{6}$$

becomes

$$axy^2 + bx^2z + cy^3 + dxz^2 + exyz = 0.$$

This has an inflexion at $(0, 0, 1)$ with $x = 0$ as inflexion tangent. In both cases, the final cubic has rational coefficients.*

6. In view of the preceding we may henceforth assume that the equation

$$f(x, y, z) = 0 \tag{1}$$

is a homogeneous cubic with rational coefficients with a known rational solution and that this solution corresponds to an inflexion point, A_0, of the cubic curve C. By choosing the inflexion tangent for a new x-axis, any other line through A_0 for the y-axis, and the harmonic line of A_0 for the z-axis, a transformation with rational coefficients,† we get Newton's form of the cubic, viz.,

$$axz^2 + by^3 + cy^2x + dyx^2 + dx^3 = 0. \tag{7}$$

---

* That k^2 is invariant under linear transformation is well known. That it is also invariant under the Cremona transformations here employed is easily verified by the method of computation described in § 7.

† Since A_0 is a rational inflexion point, its quadratic polar will factor into two linear factors one of which will correspond to the inflexion tangent through A_0 and the second to a line not through A_0 called the harmonic line of A_0. With our present coördinate system the first factor representing the inflexion tangent has rational coefficients, hence the function corresponding to the harmonic line will also have rational coefficients. Cf. Salmon's *Higher Plane Curves*, 3d edition, 1879, p. 147.

On dividing by a, which obviously can not be zero, this becomes

$$xz^2 - g(y^3 + hy^2x + myx^2 + nx^3) = 0.$$

If the three roots of

$$y^3 + hy^2x + myx^2 + nx^3 = 0$$

are designated by r_1, r_2, r_3 each line $y - r_ix = 0$ is tangent to C at $z = 0$, whence the anharmonic ratio of the cubic is that of the pencil of tangents

$$x = 0,\ y - r_1x = 0,\ y - r_2x = 0,\ y - r_3x = 0,$$

and this is $(\infty r_3 r_2 r_1) = (r_3 - r_2)/(r_3 - r_1)$. If $r_1 + r_2 + r_3 = 3r_3$, this ratio is -1 and the cubic is harmonic. Geometrically this means that either (a) A_0 is the tangential of three real points all rational or one rational, whence the cubic consists of two branches, — an odd branch infinite in extent and an even branch or oval, or (b) one tangent from A_0 is real and the other two are imaginary, whence the cubic has only one real branch, the infinite one. Thus for all rational values of $k^2 = (\infty r_3 r_2 r_1)$ other than $k^2 = -1, 2, \frac{1}{2}, 0$ there will be three distinct real tangents from (0, 0, 1) to the cubic and the oval of the cubic will consist of real points.

In what follows we shall mean by C a cubic of genus one with rational anharmonic ratio different from $-1, 2, \frac{1}{2}, 0$ and for definiteness we assume $r_1 < r_2 < r_3$. Then k^2 will be a positive rational number less than unity.

7. The above discussion has shown that if a given cubic is transformed by the indicated steps to the form (7) the value of k^2 may be found at once. We shall show that for certain values of k^2 there will be on the curve four and no more rational points in a complete group, for certain other values of k^2 there will be exactly eight such points, while the presence of a ninth rational point assures an infinite number. It will then in general be desirable to have a method for computing k^2 without reducing (1) to canonical form, for since k^2 is invariant under the transformations we are using it will usually be simpler to compute k^2 from (1) for any particular cubic studied to see if it falls into one of the categories about which we can make definite statements.

Computation of k^2. The biquadratic which gives the four tangents from the known rational point, $A_0(x_0, y_0, z_0)$ not here assumed to be an inflexion, is given by*

* Salmon's *Higher Plane Curves*, 3d edition, 1879, p. 62.

$$\Delta^2 - 4\Delta' f = 0 \tag{8}$$

where $\Delta = x_0 f_x + y_0 f_y + z_0 f_z$ and $\Delta' = x f_{x_0} + y f_{y_0} + z f_{z_0}$. By solving (8) with a convenient one of the coördinate axes, we get the binary quadratic which gives the ratios of the four points of intersection of this axis with the four tangents which, by Pappus' theorem, gives the quantity we seek. The six values of the anharmonic ratio are the negatives of the ratios of the roots of

$$z^3 - 6iz - d = 0,$$

where $i = 2(a_0 a_4 - 4a_1 a_3 + 3a_2^2)$, $d^2 = 32(i^3 - 6j^2)$, and

$$j = 6 \begin{vmatrix} a_0 & a_1 & a_2 \\ a_1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \end{vmatrix},$$

i and j being the invariants of the biquadratic whose coefficients are designated by $a_0, 4a_1, \cdots$, etc. If $i = 0$, $j \neq 0$, we will have $k^2 = -\omega$, where $\omega^3 = 1$, and the cubic is called *equianharmonic*. It has only one real branch. If $j = 0$, the cubic is *harmonic*.*

By applying to (7) the transformation

$$\begin{aligned} y' &= \sigma(y - r_3 x), \\ y' - x' &= \sigma(y - r_1 x), \\ y' - k^2 x' &= \sigma(y - r_2 x), \end{aligned}$$

whose inverse is

$$x : y : z :: -x'/(r_3 - r_1) : y' - r_3 x'/(r_3 - r_1) : z',$$

we get, dropping primes,

$$xz^2 - \alpha y(y - x)(y - k^2 x) = 0, \tag{9}$$

where $\alpha = g(r_3 - r_1)$. Furthermore, α may be assumed a positive integer without a square factor, since any square factor may be rationally absorbed by the z^2 and if α is negative the transformation

$$x : y : z :: -x' : y' - x' : z'$$

---

* See Salmon, loc. cit., p. 199, where however the notation is different and the plus sign which occurs in both equations should be a minus sign. See also Dickson's *Algebraic Invariants*, 1914, p. 55, and Clebsch-Lindemann, *Vorlesungen über Geometrie*, French edition, vol. 1, p. 297.

replaces (9) by

$$xz^2 + \alpha y(y-x)(y-k'^2x) = 0 \qquad k'^2 = 1-k^2,$$

i. e., k'^2 is positive and less than unity as was k^2.*

8. In the coördinate system corresponding to (9) the points of which the origin is tangential are $(1, 1, 0)$, $(1, k^2, 0)$, $(1, 0, 0)$, and as before, $0<k^2<1$. The coördinates of points (x, y, z) on C are proportional to

$$\text{(10)} \qquad \mathrm{sn}^3 u : \mathrm{sn}\, u : \sqrt{\alpha}\, \mathrm{cn}\, u\, \mathrm{dn}\, u,$$

and the elliptic arguments of the four rational points already known to be on C may be taken to be

$$\text{(11)} \qquad \alpha_0 = 0, \quad \alpha_1 = \omega'/2, \quad \alpha_2 = (\omega+\omega')/2, \quad \alpha_3 = \omega/2,$$

corresponding respectively to the points

$$A_0(0, 0, 1), \qquad A_1(1, 0, 0), \qquad A_2(1, k^2, 0), \qquad A_3(1, 1, 0),$$

where ω is a real period and ω' a pure imaginary period. We see from the figure (p. 483) that the lines $y=0$ and $y=k^2x$ are tangent to the even branch of the curve at the points with elliptic arguments $\omega'/2$ and $(\omega+\omega')/2$ respectively, while $y=x$ is tangent to the odd branch at the point with argument $\omega/2$. In fact the convention (11) and our choice of coördinate triangle enable us to represent points on the *odd* branch in terms of ω *alone* while points on the *even* branch are represented by $\lambda\omega+\omega'/2$ (λ real).†

From (9) and the last footnote we see that α is an arithmetic invariant of the cubic as was stated by B. Levi in the article cited. He further states in the paper read at the Rome Congress, page 176, that by birational transformation rational in k^2 we can always obtain the canonical form

$$xz^2 - y(y-x)(y-k^2x) = 0.$$

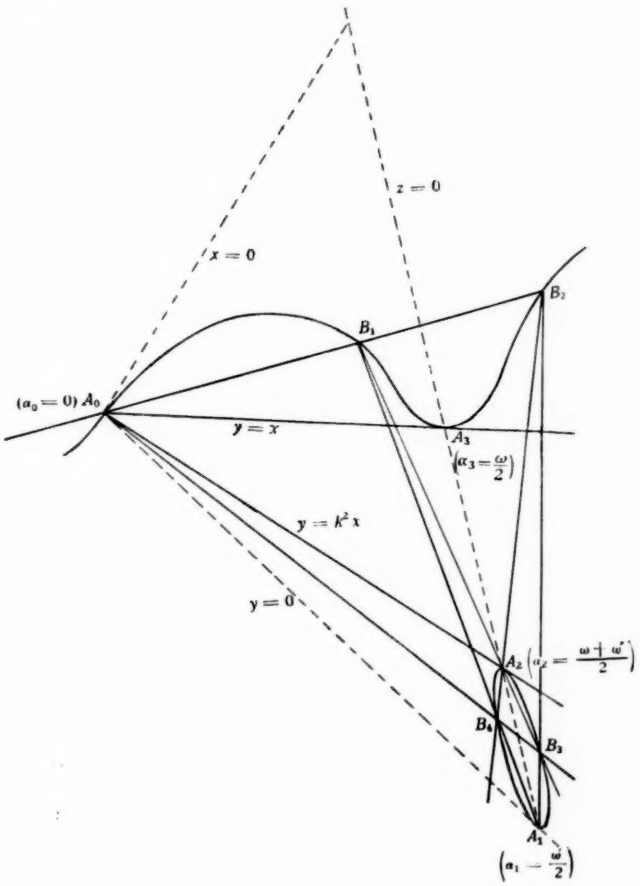
This is equivalent to the statement that two cubics with the same rational k^2 are birationally equivalent. That this is not the case is shown by the following example:

The equation $xz^2 - y(y-x)(y-3x/4) = 0$ has eight rational solutions, viz.: $(0, 0, 1)$, $(1, 0, 0)$, $(1, 1, 0)$, $(4, 3, 0)$, $(4, 6, \pm 3)$, $(4, 2, \pm 1)$,

* In fact, α is the numerical value of the largest non-square factor appearing in i^3/j^2; Levi, loc. cit., p. 751.

† Clebsch-Lindemann, loc. cit., p. 609–610.

while $xz^2 - 2y(y-x)(y-3x/4) = 0$ has only four rational solutions, the first four of the above list. Obviously the second equation is not birationally transformable into the first. This conclusion will follow also in § 9 from an analytical discussion of conditions under which the period is divisible.



We may summarize what has preceded thus:

THEOREM. *The equation $f(x, y, z) = 0$ of a plane cubic curve of genus unity, with rational anharmonic ratio, $k^2 \neq 0$, and with a known rational point, may by birational transformation with rational coefficients be brought to one of the forms*

(i) $\quad xz^2-\alpha y(y^2-yx+x^2) \quad = 0, \quad k^2 = -1, \tfrac{1}{2}, 2;$

(ii) $\quad xz^2-\alpha y(y^2+dx^2) \quad = 0, \quad k^2 = -1, \tfrac{1}{2}, 2;$

(iii) $\quad xz^2-\alpha y(y-x)(y-k^2x) = 0, \quad 0<k^2<1, k^2 \neq \tfrac{1}{2},$

where d is a rational number $\gtrless 0$, and α is a positive integer, containing no square factor. On the cubic there will be in the respective cases one, one or three, three groups of two rational points closed under the fundamental construction. Where there are three such two-groups they form a complete group of four points. The elliptic parameters of these rational points may be taken to be 0, $\omega/2$ and 0, $\omega/2$, $\omega'/2$, $(\omega+\omega')/2$ where ω and ω' represent convenient real and pure imaginary periods. The rank will be one, one or two, two.

9. Giving our attention now to the non-harmonic cubic with k^2 rational the question next arises: For what values of α and k^2 will there exist other rational points on C? If we solve

$$mx-\alpha y = 0$$

simultaneously with (9), in addition to the intersection at $A_0(0, 0, 1)$ we get the two intersections

$$\alpha : m : \pm\sqrt{m(m-\alpha)(m-k^2\alpha)}\,.$$

To find the solutions of

$$m(m-\alpha)(m-k^2\alpha) = \text{a rational square} \tag{12}$$

we note that if the product of three quantities is a square, so is the quotient of any two of them by the third. Thus we shall have

$$(13)\qquad \begin{aligned} &(1) & m(m-\alpha) &= (m-k^2\alpha)s^2,\\ &(2) & m(m-k^2\alpha) &= (m-\alpha)s^2,\\ &(3) & (m-\alpha)(m-k^2\alpha) &= ms^2, \end{aligned}$$

where α and k^2 are known constants, s is an arbitrary rational parameter, and any value of m which satisfies (12) will also satisfy the three equations (13). Thus the problem of finding all rational solutions of a cubic in m is replaced by the problem of finding all rational solutions of three quadratics in m. The m-discriminants of these equations must be squares different from zero. By making simple transformations the

discriminants may be reduced to the Pell equation $x^2 - Dy^2 = 1$ which will be satisfied in the respective cases by

$$(14)\quad \begin{aligned} &(1)\quad k^2 = \frac{(\alpha+s^2)^2-c^2}{4\alpha s^2}, && m = \frac{(\alpha+s^2)\pm c}{2},\\ &(2)\quad k^2 = \frac{(\alpha-c)s^2+c^2}{\alpha c}, && m = c \text{ or } \alpha s^2/c,\\ &(3)\quad k^2 = \frac{2(\alpha-s^2)}{\alpha}, && m = 2\alpha \text{ or } (\alpha-s^2), \end{aligned}$$

where s and c are arbitrary rational parameters, $\alpha > 0$ is an integer, k^2 is rational, positive and less than one.*

Each value of k^2 in (14) gives two distinct values for m and each value of m gives the two rational points on the curve

$$(15)\qquad \alpha : m : \pm\sqrt{m(m-\alpha)(m-k^2\alpha)}.$$

For the respective values of k^2 these points will have for coördinates (the upper subscript is to be read with the upper sign):

$$\begin{aligned} &\text{(i)} && B_{\substack{1\\2}} && 4\alpha s : 2(\alpha+s^2+c)s : \pm(\alpha+s^2+c)(-\alpha+s^2+c),\\ & && B_{\substack{3\\4}} && 4\alpha s : 2(\alpha+s^2-c)s : \mp(\alpha+s^2-c)(\alpha-s^2+c);\\ &\text{(ii)} && B_{\substack{1\\2}} && \alpha : c : \pm s(c-\alpha),\\ & && B_{\substack{3\\4}} && c : s^2 : \pm s(s^2-c);\\ &\text{(iii)} && B_{\substack{1\\2}} && 1 : 2 : \pm 2s,\\ & && B_{\substack{3\\4}} && \alpha : \alpha-s^2 : \pm s(\alpha-s^2). \end{aligned}$$

* We may look upon the last paragraph thus: The normal form (9) of the cubic has two parameters, α and k^2. We seek all values k^2 which for a given α will assure the presence of a fifth rational point on the curve. Obviously, k^2 will be a function of α. If for any one of the equations (13) we could find all values of k^2 which would make the discriminant a positive square it would be unnecessary to consider the two remaining equations. However the way in which the parameters enter these discriminants has thus far made it impossible to be certain that all such expressions have been obtained, hence we use all three equations (13). Even so, no assertion is made as to the completeness with which the problem is solved, but it is true that if k^2 is expressible in any of the three forms (14), the cubic C in addition to the four rational points already found will have a fifth rational point, in fact will have at least four more rational points, as is shown in the next paragraph.

We next compute the tangentials of B_1, B_2, B_3, B_4 to see if, and under what conditions, the tangentials may coincide with points already found thus assuring that the group of eight points may be a complete group. In every case it is found that B_1 and B_4 have the same point for tangential, and that B_2 and B_3 have the same point for tangential. Calling these points B_{14} and B_{23} respectively, they are

$$\text{(i)} \quad B_{\substack{14\\23}} \quad 8\alpha s^3 : 2(\alpha + s^2)^2 s : \pm c(\alpha + s^2)(\alpha - s^2),$$

$$\text{(ii)} \quad B_{\substack{14\\23}} \quad 8\alpha s^3 c^3 : 2sc(s^2\alpha + c^2)^2 : \pm(s^2\alpha + c^2)(s^2\alpha - c^2)(2s^2c - s^2\alpha - c^2),$$

$$\text{(iii)} \quad B_{\substack{14\\23}} \quad 8s^3\alpha : 2s(s^2 + \alpha)^2 : \pm(s^2 + \alpha)(s^2 - \alpha)(3s^2 - \alpha).$$

Now since A_0, B_1, and B_2 are collinear; since A_0, B_3, and B_4 are collinear; and since A_0, B_{14}, and B_{23} are collinear, the only way in which B_{14} and B_{23} could coincide with any of the points previously found (thus completing an eight-group), would be for them to coincide simultaneously with A_1, or with A_2, or with A_3, each of which has the z-coördinate zero. Imposing this condition and remembering the previously assumed restrictions on α and k^2 we find that for $\alpha = 1$ in the first and second cases B_{14} and B_{23} coincide with A_3; k^2 is then $1 - c^2/4$ and $2c - c^2$ respectively, where c is any rational number with modulus < 2. For no other value of α is the eight-group closed. We state the

THEOREM. *If the numerical value of i^3/j^2 is a square, i.e., if i is a square and if one value of the anharmonic ratio of the cubic is a square less than unity, there will be on the cubic a complete group of eight points.*

If the invariants of the cubic are designated by S and T, from the relations $i^3/j^2 = -2^7 \cdot 3 \cdot S^3/T^2$ we can state the theorem thus:

THEOREM. *If the invariant S of the ternary cubic form $f(x, y, z) = 0$ is six times a square and if the cubic curve represented by $f = 0$ has anharmonic ratio one value of which is a positive rational square < 1, the curve will have a complete group of eight rational points.*

For all values of $\alpha \neq 1$ and values of k^2 given by (14) the preceding shows that there will be at least two additional rational points on C obtainable by the fundamental construction from the eight tabulated.

10. A study of the elliptic arguments of the points on the cubic furnishes the same *sufficient* conditions for the presence on the curve of a complete four-group or of a complete eight-group, together with the general

THEOREM. *If the tangential of the point B_1 coincides with A_3 there will be a complete eight-group on the cubic while in the contrary case there will be an infinite number of rational points on the cubic; in both cases all the*

rational points mentioned may be obtained from two of the group, one of A_1, A_2 and one of B_1, B_2, B_3, B_4. The rank is two.

In other words, if the elliptic argument corresponding to any one of the B's is an aliquot part of a primitive period* of the cubic, it will be one fourth of it, and the four B's will complete an eight-group, while if the elliptic parameter of B_1, say, does not divide a primitive period by four it will not divide it rationally and there will be an infinite number of rational points obtainable from these two by the fundamental construction. Proof of this theorem is in § 11.

The periods of the Jacobi elliptic functions are given by

$$4mK + 4m'iK',$$

where m and m' are positive or negative integers. Moreover $2K + 2iK'$ is a primitive period for the cubic, since an increase of u by $2K + 2iK'$ changes the sign of each of the three coördinates

$$\varrho\,\mathrm{sn}^3 u, \quad \varrho\,\mathrm{sn}\,u, \quad \varrho\sqrt{a}\,\mathrm{cn}\,u\,\mathrm{dn}\,u.$$

Using the notation $\omega = 2K$, $\omega' = 2iK'$, and calling $\alpha_0, \alpha_1, \cdots, \beta_1, \beta_2, \cdots$ the arguments of $A_0, A_1, \cdots, B_1, B_2, \cdots$, we can by easy computation find that we may take

$$\alpha_0 = 0, \quad \alpha_1 = \omega'/2, \quad \alpha_2 = (\omega + \omega')/2, \quad \alpha_3 = \omega/2.$$

Since the sum of the elliptic arguments of three collinear points is congruent to zero modulo a period, if A_3 is to be the tangential of one and consequently of all four of the B's, the elliptic arguments of the latter points must be

(16) $\pm\frac{1}{4}\omega = \pm\frac{1}{2}K$ and $\pm\frac{1}{4}\omega + \omega'/2 = \pm\frac{1}{2}K + iK'$ (mod a period).

These are found to be (Cayley, loc. cit.)

(17) $1 : (1+k') : \pm\sqrt{a}\,k'(1+k')$ and $1 : (1-k') : \mp\sqrt{a}\,k'(1-k')$,

where k'^2 is the modulus conjugate to k^2. It is clear that these four ratios are real and rational if and only if k'^2 is a positive square and $a = 1$. Referring to (14) and the results near the close of § 9, we find that k'^2 had the values $c^2/4$ and $(c-1)^2$, $|c| < 2$. Using these values

* See Cayley, *Elliptic Functions*, 2d edition, 1895, pp. 70 and 74.

in (16) we get numbers proportional to the coördinates of B_1, B_2, B_3, B_4 as given in the first tabulation after (15), as was to be expected. In any case the points whose arguments are given by (16) are on the cubic, since they satisfy (9) identically, but only if (1) $s^2 = \alpha = 1$, $k^2 = (4-c^2)/4$, and (2) $\alpha = c^2/s^2 = 1$, $c = \pm s$, $k^2 = 2c - c^2$ will these points be rational and have A_3 as tangential, i. e. complete an eight-group.

11. We next enquire whether and under what conditions a group of points consisting of nine or more may close. In this case neither B_{14} nor B_{23} coincides with A_3. We construct the tangential of B_{14}, say. Call it N with argument ν;

$$\nu = -2\beta_{14} = 4\beta_1.$$

There are two cases to be considered:

(a) $\nu = K$, i. e. N coincides with A_3;

(b) $\nu \neq K$ and the tangential of N may be constructed.

In case (a),

$$\operatorname{sn}^2\beta_1 = \operatorname{sn}^2 \tfrac{1}{4} K = \frac{\sqrt{1+k'} - \sqrt{k'}}{\sqrt{1+k'}\,[1+\sqrt{k'}]}.$$

Now if the ratios $\operatorname{sn}^3\nu : \operatorname{sn}\nu : \sqrt{\alpha}\operatorname{cn}\nu\operatorname{dn}\nu$ are to be rational it will follow that $\operatorname{sn}^2\nu$ and $\sqrt{\alpha}\operatorname{sn}\nu\operatorname{cn}\nu\operatorname{dn}\nu$ must each be rational. Imposing these two conditions on ν we get for case (a) that $\nu = \frac{1}{4}K$ is a rational point if and only if

(1) k'^2 = positive rational square,

(2) $\alpha = 1$,

(3) $1 + k'$ = rational square.

But (1) and (2) are sufficient to assure a closed group of *eight* points. Hence, if B_{14} does not coincide with A_3 neither can its tangential, N.

In case (b), an easy induction shows that if B_{14} does not coincide with A_3, no point obtained by constructing successive tangentials from B_{14} can do so. The conditions will always include the ones obtained for case (a) above. In fact, the elliptic arguments of these successive tangentials, all obviously on the odd branch of the curve, will be $\pm 2^s\beta_1$, where $s = 1, 2, 3, \cdots$. If the sequence of points terminates in an inflexion, $\pm\frac{2}{3}K$, one of the points will have argument $\frac{1}{3}K$. The proof that these two points are not rational points on the cubic (9) is given in § 12. Also, if the sequence of points closes forming a polygon we shall have (see (4a), § 4)

$$2^s\beta_1 = \pm\frac{1}{2^t\cdot 3\cdot M}K,$$

consequently some rational point of the group will have argument $(1/3M)K$.

We shall first show that in this case also the point with argument $\frac{1}{3}K$ must be in the group. For we can find positive integers $r, s, t, \cdots$ so that

$$M = 2^r \pm 2^s \pm 2^t \pm \cdots + 1$$

(see (4) in § 4), whence unless $M = 3$, by tangentials and secants we get that the rational point with argument

$$\beta = \frac{2^r \pm 2^s \pm 2^t \pm \cdots + 1}{3 \cdot M} K \tag{18}$$

is one of the points belonging to the complete group containing the vertices of the polygon. The exception occurs if $M = 3$, since in this case only the three vertices of the triangle, with arguments $\frac{2}{9}K$, $\frac{8}{9}K$, $\frac{14}{9}K$ (note: none of these is an inflexion) are obtained by tangentials and no additional points are obtained by secants. The proof is then complete with the following theorem:

12. *On the cubic* (9) *the real inflexion points with arguments* $\pm\frac{2}{3}K$ *are not rational points.*

Proof. By (18), $\beta = \frac{1}{3}K$ will be a rational point. It is easy to verify that the following quantities are rational: $\operatorname{sn}^2\beta$, $\sqrt{\alpha}\operatorname{sn}\beta\operatorname{cn}\beta\operatorname{dn}\beta$, $\operatorname{cn}2\beta$, $\operatorname{dn}2\beta$, $\sqrt{\alpha}\operatorname{sn}2\beta$, $\operatorname{sn}^2 2\beta$. Hence

$$1 = \operatorname{sn}(2\beta+\beta) = \frac{\operatorname{sn}2\beta\operatorname{cn}\beta\operatorname{dn}\beta + \operatorname{sn}\beta\operatorname{cn}2\beta\operatorname{dn}2\beta}{1 - k^2\operatorname{sn}^2 2\beta\operatorname{sn}^2\beta}.$$

Now the denominator is rational, but the numerator is not unless $\alpha = 1$, which is contrary to the present hypothesis. Hence β is not a rational point. Then 2β can not be the argument of a rational point since β, 2β, and $3\beta = K$ are collinear and K is the argument of $A_3(1, 1, 0)$.

13. **Recapitulation.** (*a*) Every cubic equation of genus one in three homogeneous variables having rational coefficients, rational anharmonic ratio, and known to have one rational solution may by a proper choice of a coördinate triangle be reduced to the form

$$xz^2 - \alpha y(y-x)(y-k^2x) = 0, \tag{9}$$

where k^2 is a positive proper fraction and α a positive integer not containing a square factor.

(*b*) On the cubic C, the graph of (9), there will always be four rational points,

$$A_0(0, 0, 1), \quad A_1(1, 0, 0), \quad A_2(1, k^2, 0), \quad A_3(1, 1, 0),$$

with elliptic arguments

$$0, \quad iK', \quad K+iK', \quad K.$$

This is a complete four-group with rank two since two of A_1, A_2, and A_3 must be given before the remaining two can be constructed. It is made up of three complete two-groups, A_0 and A_i $(i=1, 2, 3)$, each of rank one.

(*c*) If $\alpha = 1$, k^2 is expressible as in (14_1) or (14_2) with $1-k^2$ equal to a positive square, there will be on C four additional rational points, B_i $(i = 1, 2, 3, 4)$, each having A_3 for its tangential and lying by twos on lines through A_0, by twos on lines through A_1, and by twos on lines through A_2. The rank of this eight-group is two. The elliptic arguments of the points B_i will be

$$\tfrac{1}{2}K, \quad -\tfrac{1}{2}K, \quad \tfrac{1}{2}K+iK', \quad -\tfrac{1}{2}K+iK'.$$

Their rational coördinates are given by (16). All the points on the odd branch of the curve are expressible in terms of the real (primitive) period $\omega = 2K$, while points on the even branch will have arguments $\lambda\omega+\frac{1}{2}\omega'$, $\omega' = 2iK'$ (λ real).

(*d*) If k^2 is expressible in terms of α and of the arbitrary rational parameters s and c, in one of the forms

$$\frac{(\alpha+s^2)^2-c^2}{4\alpha s^2}, \quad \frac{(\alpha-c)s^2+c^2}{\alpha c}, \quad \frac{2(\alpha-s^2)}{\alpha}$$

with not simultaneously $\alpha=1$ and $(1-k^2)$ a square, there will be on C the eight rational points already listed and in addition an infinity of rational points obtainable from these by the fundamental construction.

THE UNIVERSITY OF CHICAGO,
CHICAGO, ILL.

ON THE EXISTENCE OF THE STIELTJES INTEGRAL*

BY

H. L. SMITH

Lebesgue† has shown how to treat the Riemann integral by studying the content of an associated point set. The present paper is an attempt in the same direction for the Stieltjes integral. A pair of conditions are found which are necessary and sufficient for the existence of the integral, one of which concerns itself with the associated point set. The other is automatically satisfied for a large class of integrals comprising (1) those for which the associated point set is a (continuous) curve with at most a finite number of multiple points; and (2) those for which the measure function is of limited variation. A consequence is that a simple closed curve must be squarable if its line integral $\int y\,dx$ exists. Among the examples given is one which shows that a simple closed curve may be squarable and still fail to have an existent line integral $\int y\,dx$.

1. **Definitions and notations.** If I', I'' are two sub-intervals of T, $I'\cdot I''$ will denote the interval common to I', I''.

A general partition $t_0 = 0 < t_1 < \cdots < t_{n-1} < t_n = 1$ of the interval T: $0 \leqq t \leqq 1$ will be denoted by the notation π; a partition of a sub-interval I of T will be denoted by $\pi(I)$. A general cell $t_i \leqq t \leqq t_{i+1}$ of a partition π will be denoted by $\Delta\pi$. The symbol $\sum_{\Delta\pi}^{\pi}$ will denote a summation over all the cells $\Delta\pi$ of a partition π. The upper (greater) and lower (lesser) end points of an interval I will be denoted by $\overline{I}$, $\underline{I}$ respectively, and t^I will denote any point of an interval I. By $\pi'\times\pi''$ will be denoted the partition consisting of all non-singular $\Delta\pi'\cdot\Delta\pi''$.

Every numerically-valued function $\theta(t)$ defined on T gives rise to an associated function $\theta(I)$ on the class of all sub-intervals I of T defined by the equation

$$\theta(I) \equiv \theta(\overline{I}) - \theta(\underline{I}).$$

The symbols $S_\pi\psi\Delta\varphi$ and $S_\pi^0\,\psi\Delta\varphi$ are defined by the equations

$$S_\pi\psi\Delta\varphi \equiv \sum_{\Delta\pi}^{\pi}\psi(t^{\Delta\pi})\varphi(\Delta\pi),$$

$$S_\pi^0\psi\Delta\varphi \equiv \sum_{\Delta\pi}^{\pi}\frac{1}{2}\{\psi(\underline{\Delta\pi}) + \psi(\overline{\Delta\pi})\}\varphi(\Delta\pi);$$

* Presented to the Society, December 29, 1923.

† *Leçons sur l'Intégration*, 1904, p. 45.

and are respectively multiply-valued and singly-valued functions of π for given φ, ψ.

A partition π_1 is *finer* than a partition π_2, in notation $\pi_1 \mathrm{F} \pi_2$, if every cell $\Delta\pi_1$ of π_1 lies entirely in some cell $\Delta\pi_2$ of π_2. The binary relation F is transitive, reflexive and has the composition property as defined by E. H. Moore and the author.* It, therefore, serves to define a limit process $\mathrm{L_F}$ applicable to numerically-valued functions $\theta(\pi)$ defined for the class of all partitions of T. Thus $\mathrm{L_F}\,\theta = a$ provided there exists a system $(\pi_e \mid e)$ $(e > 0)$ such that

$$|\theta(\pi) - a| \leqq e \qquad (\pi \mathrm{F} \pi_e) \qquad (e > 0).$$

Every partition π has a norm $\mathrm{N}\pi$ defined as the length $\overline{\Delta\pi} - \underline{\Delta\pi}$ of its longest cell $\Delta\pi$. This gives rise to a second limit process $\mathrm{L_N}$ on functions $\theta(\pi)$ defined as follows: $\mathrm{L_N}\theta = a$ provided there exists a system $(d_e \mid e)$ such that

$$|\theta(\pi) - a| \leqq e \qquad (\pi,\ \mathrm{N}\pi \leqq d_e) \qquad (e).$$

The (Riemann) Stieltjes integral $\int_0^1 \psi(t)\, d\varphi(t)$ or more briefly $\int_0^1 \psi\, d\varphi$ exists in the sense

$$\begin{matrix} (FS) \\ (FW) \\ (NS) \\ (NW) \end{matrix}$$

provided

$$\begin{matrix} \mathrm{L_F}\, S_\pi\, \psi\Delta\varphi \\ \mathrm{L_F}\, S_\pi^0\, \psi\Delta\varphi \\ \mathrm{L_N}\, S\ \ \psi\Delta\varphi \\ \mathrm{L_N}\, S^0\, \psi\Delta\varphi \end{matrix}$$

exists. The same symbol may be used in each of these cases, since, if the integral exists simultaneously in two or more senses, the values are the same. This is easily seen from the following.

Obviously if the integral $\int_0^1 \psi\, d\varphi$ exists in the sense $\frac{FS}{NS}$ then it exists (with the same value) in the sense $\frac{FW}{NW}$, which justifies the notations S (= strong) and W (= weak). Moreover if the integral exists in the sense $\frac{NW}{NS}$ then it exists also in the sense $\frac{FW}{FS}$.

* E. H. Moore and H. L. Smith, American Journal of Mathematics, vol. 44 (1922), p. 104.

2. **A necessary and sufficient condition.** The *oscillation* of $S\psi\Delta\varphi$ on I, in notation $O_I S\psi\Delta\varphi$, is defined as the least upper bound of $|S_{\pi'(I)}\psi\Delta\varphi - S_{\pi''(I)}\psi\Delta\varphi|$ for all partitions $\pi'(I)$, $\pi''(I)$ of I (I in T); and $O_I S^0\psi\Delta\varphi$ is similarly defined as the least upper bound of $|S^0_{\pi'(I)}\psi\Delta\varphi - S^0_{\pi''(I)}\psi\Delta\varphi|$. The symbols $O_\pi S\psi\Delta\varphi$, $O_\pi S^0\psi\Delta\varphi$ are defined by the equations

$$O_\pi S\psi\Delta\varphi \equiv \sum_{\Delta\pi}^{\pi} O_{\Delta\pi} S\psi\Delta\varphi,$$

$$O_\pi S^0\psi\Delta\varphi \equiv \sum_{\Delta\pi}^{\pi} O_{\Delta\pi} S^0\psi\Delta\varphi.$$

THEOREM I. *In order that* $\int_0^1 \psi\, d\varphi$ *exist in the sense* (FS), (FW), (NS) *or* (NW) *it is necessary and sufficient that* $\underset{\pi}{\mathrm{L}_{\mathrm{F}}}\, O_\pi S\psi\Delta\varphi = 0$, $\underset{\pi}{\mathrm{L}_{\mathrm{F}}}\, O_\pi S^0\psi\Delta\varphi = 0$, $\underset{\pi}{\mathrm{L}_{\mathrm{N}}}\, O_\pi S\psi\Delta\varphi = 0$ *or* $\underset{\pi}{\mathrm{L}_{N}}\, O_\pi S^0\psi\Delta\varphi = 0$ *respectively.*

We prove the theorem for the sense (FS).

The condition is *necessary*. For let π be any partition of T and e any positive number. Then there are two partitions $\pi'\,\mathrm{F}\,\pi$, $\pi''\,\mathrm{F}\,\pi$ such that

$$0 \leqq O_{\Delta\pi} S\psi\Delta\varphi - \{S_{\pi'(\Delta\pi)}\psi\Delta\varphi - S_{\pi''(\Delta\pi)}\psi\Delta\varphi\} \leqq \frac{e}{n},$$

where $\pi'(\Delta\pi)$, $\pi''(\Delta\pi)$ denote respectively the partitions π', π'' as on $\Delta\pi$ and n is the number of cells in π. Hence

$$0 \leqq S_{\pi'}\psi\Delta\varphi - S_{\pi''}\psi\Delta\varphi \leqq O_\pi S\psi\Delta\varphi \leqq e + S_{\pi'}\psi\Delta\varphi - S_{\pi''}\psi\Delta\varphi.$$

On applying the quasi-limit* $\underset{\pi}{L_{\mathrm{F}}}$ to this inequality, there results

$$0 \leqq \underset{\pi}{L_{\mathrm{F}}}\, O_\pi S\psi\Delta\varphi \leqq e \qquad (e)$$

or

$$\underset{\pi}{L_{\mathrm{F}}}\, O_\pi S\psi\Delta\varphi = 0,$$

uniquely; which proves the necessity.†

The condition is *sufficient*. For if $\pi'''\,\mathrm{F}\,\pi'$, $\pi'''\,\mathrm{F}\,\pi''$, then

$$\begin{aligned}|S_{\pi'}\psi\Delta\varphi - S_{\pi''}\psi\Delta\varphi| &\\ \leqq |S_{\pi'}\psi\Delta\varphi - S_{\pi'''}\psi\Delta\varphi| &+ |S_{\pi'''}\psi\Delta\varphi - S_{\pi''}\psi\Delta\varphi| \\ \leqq O_{\pi'}\psi\Delta\varphi &+ O_{\pi''}\psi\Delta\varphi;\end{aligned}$$

---

* Moore and Smith, loc, cit., p. 110.

† Moore and Smith, loc. cit., p. 112, Theorem 8.

from which the sufficiency follows on taking the simultaneous F-limit as to π' and π''.

3. **Some necessary conditions.** $O_I\,\psi$, the *oscillation* of ψ on I, is defined as the least upper bound of

$$|\psi(t_1^I)-\psi(t_2^I)|$$

for all t_1^I, t_2^I.

INEQUALITY A. $\quad O_I\,S\,\psi\,\Delta\varphi \geqq \quad (O_I\,\varphi)\,|\varphi(I)|$.

INEQUALITY A^0. $\quad O_I\,S^0\,\psi\,\Delta\varphi \geqq \frac{1}{2}(O_I\,\psi)\,|\varphi(I)|$.

We give the (slightly) more difficult proof, that of A^0. Let $\varphi(I) \neq 0$; the inequality is obvious otherwise. For every $e(>0)$ take t' and t'' in I such that

$$[\psi(t')-\psi(t'')]\ \mathrm{sgn}\,\varphi(I) \geqq O_I\,\psi - e,$$

so that

$$[\psi(t')-\psi(t'')]\,\varphi(I) \qquad \geqq [O_I\,\psi - e]\,|\varphi(I)|\,.$$

Then

$$\frac{1}{2}[\psi(t')-\psi(t'')]\,\varphi(I) = S_{\pi'(I)}\,\psi\,\Delta\varphi - S_{\pi''(I)}\,\psi\,\Delta\varphi,$$

where

$$S_{\pi'(I)}\,\psi\,\Delta\varphi = \frac{\psi(\underline{I})+\psi(t')}{2}[\varphi(t')-\varphi(\underline{I})] + \frac{\psi(t')+\psi(\bar{I})}{2}[\varphi(\bar{I})-\varphi(t')],$$

$$S_{\pi''(I)}\,\psi\,\Delta\varphi = \frac{\psi(\underline{I})+\psi(t'')}{2}[\varphi(t'')-\varphi(\underline{I})] + \frac{\psi(t'')+\psi(\bar{I})}{2}[\varphi(\bar{I})-\varphi(t'')];$$

so that

$$S_{\pi'(I)}\,\psi\,\Delta\varphi - S_{\pi''(I)}\,\psi\,\Delta\varphi \geqq \frac{1}{2}(O_I\,\psi)\,|\varphi(I)| - \frac{1}{2}e\,|\varphi(I)| \qquad (e).$$

From this inequality A^0 follows.

THEOREM N 1. *In order that* $\int_0^1 \psi\,d\varphi$ *exist in the sense* (FS) *or* (FW) *it is necessary that*

$$\underset{\pi}{\mathrm{L_F}} \sum_{\Delta\pi}^{\pi} (O_{\Delta\pi}\,\psi)\,|\varphi(\Delta\pi)| = 0;$$

in the sense (NS), (NW) *it is necessary that*

$$\underset{\pi}{\mathrm{L_N}} \sum_{\Delta\pi}^{\pi} (O_{\Delta\pi}\,\psi)\,|\varphi(\Delta\pi)| = 0.$$

This theorem follows at once from inequalities A, A^0.

THEOREM N2. *In order that* $\int_0^1 \psi\, d\varphi$ *exist in the sense* (NS) *or* (NW) *it is necessary that* φ, ψ *have no simultaneous discontinuities on* T; *in the sense* (FS) *that* φ, ψ *have no simultaneous right-hand discontinuities or simultaneous left-hand discontinuities.*

The proof of this theorem is not difficult and is omitted.

4. **On the independence of the four senses** (FS), (FW), (NS), (NW). Of the eleven situations as to the simultaneous existence of $\int_0^1 \psi\, d\varphi$ in different senses indicated in Table 0 the first four are excluded by the fact that (NS) implies (FS); the next four by the fact that (FS) implies (FW); the ninth by the fact that (NS) implies (NW); and the tenth by the fact that (NW) implies (FW). The last one is excluded by Theorem N2 and Theorem J.

Table 0

	FS	FW	NS	NW
(1)	−	+	+	+
(2)	−	+	+	−
(3)	−	−	+	−
(4)	−	−	+	+
(5)	+	−	+	+
(6)	+	−	+	−
(7)	+	−	−	+
(8)	+	−	−	−
(9)	+	+	+	−
(10)	−	−	−	+
(11)	+	+	−	+

THEOREM J. *If* φ, ψ *have no simultaneous discontinuities, then the existences of* $\int_0^1 \psi\, d\varphi$ *in the senses* (FS) *or* (FW) *imply existences in the respective senses* (NS) *or* (NW).

We prove the theorem for the strong senses. We note first that

$$\underset{\pi_0}{\mathrm{L}_{\mathrm{N}}} [S_{\pi_0} \psi \Delta \varphi - S_{\pi \times \pi_0} \psi \Delta \varphi] = 0 \qquad (\pi).$$

Hence

$$(\underset{\pi}{\mathrm{L}_{\mathrm{F}}} \underset{\pi_0}{\mathrm{L}_{\mathrm{N}}}) [S_{\pi_0} \psi \Delta \varphi - S_{\pi \times \pi_0} \psi \Delta \varphi] = 0.$$

Also

$$(\underset{\pi}{\mathrm{L}_{\mathrm{F}}} \underset{\pi_0}{\mathrm{L}_{\mathrm{N}}}) S_{\pi \times \pi_0} \psi \Delta \varphi = \int_0^1 \psi\, d\varphi.$$

Therefore

$$\int_0^1 \psi\, d\varphi = (\underset{\pi}{\mathrm{L}}{}_{\mathrm{F}}\, \underset{\pi_0}{\mathrm{L}}{}_{\mathrm{N}})\, S_{\pi_0}\, \psi\, \Delta\varphi = \underset{\pi_0}{\mathrm{L}}{}_{\mathrm{N}}\, S_{\pi_0}\, \psi\, \Delta\varphi, \qquad \text{Q. E. D.}$$

There remain of the sixteen possible cases only the five exhibited in Table 1. We now give examples to show that these situations actually occur.

Table 1

	FS	FW	NS	NW
I	+	+	+	+
II	+	+	—	—
III	—	+	—	+
IV	—	+	—	—
V	—	—	—	—

Where two examples are given the second (more complicated) one is such that the situation in question holds not only for T but for every I in T.

I. $\varphi(t) = 0, \quad \psi(t) = 0, \quad 0 \leqq t \leqq 1.$

II. $\varphi(t) = 0, \quad 0 \leqq t \leqq \frac{1}{2}; \quad \varphi(t) = 1, \quad \frac{1}{2} < t \leqq 1;$
$\psi(t) = 0, \quad 0 \leqq t < \frac{1}{2}; \quad \psi(t) = 1, \quad \frac{1}{2} \leqq t \leqq 1.$

II′. $\varphi(t) = \sum_{n=1}^{\infty} a_n \epsilon'_n(t), \qquad \psi(t) = \sum_{n=1}^{\infty} a_n \epsilon''_n(t),$

where $a_n = (1/9^{n-1})$ $(n = 1, 2, 3, \cdots)$ and $\epsilon'_n(t) = 1$ when t satisfies one of the inequalities $(3k+1)/3^n \leqq t < (3k+2)/3^n$ $(k = 0, \cdots, 3^{n-1}-1)$, $= 0$ otherwise; and $\epsilon''_n(t) = 1$ when t satisfies one of the inequalities $(3k+1)/3^n < t \leqq (3k+2)/3^n$ $(k = 0, \cdots, 3^{n-1}-1)$, $= 0$ otherwise.

III. $\varphi(t) = \psi(t) = 0, \quad 0 \leqq t < 1; \quad \varphi(1) = \psi(1) = 1.$

III′. $\psi(t) = \varphi(t)$ where $\varphi(t)$ is the parametric representation of the x-coördinate of the Peano-Moore space filling curve as given by E. H. Moore (these Transactions, vol. 1 (1900), p. 80, eq. 27).

IV. $\varphi(t) = \psi(t) = 0, \quad 0 \leqq t \leqq 1, \quad t \neq \frac{1}{2};$
$\varphi(\frac{1}{2}) = \psi(\frac{1}{2}) = 1.$

IV′. $\varphi\left(\frac{2k+1}{2^m}\right) = \psi\left(\frac{2k+1}{2^m}\right) = \frac{1}{4^{m-1}} \qquad (k = 0, \cdots, 2^{m-1}-1);$

$\varphi(t) = \psi(t) = 0, \quad t$ not of form $\frac{2k+1}{2^m}$.

V. $\varphi(t) = \psi(t) = 0$, t rational; $\quad \varphi(t) = \psi(t) = 1$, t irrational.

5. Some lemmas on the operator E. Convex sets. Let α denote a planar set of points. By $\mathrm{E}(\alpha)$ will be denoted the set of all points on closed segments joining pairs of points of α. $\mathrm{E}^2(\alpha) \equiv \mathrm{E}\{\mathrm{E}(\alpha)\}$, etc. If

$E(\alpha) = \alpha$, the set is *convex*. Concerning the operation E the following simple propositions hold:

E 1. If α is connected, E (α) is convex.
E 2. If α is any set, E (α) is connected.
E 3. If α is any set, $E^2(\alpha)$ is convex.
E 4. If α is any set, $E^2(\alpha)$ is the least convex super set of α.
E 5. If α is bounded, E (α) is bounded.
E 6. If α is bounded and closed, E (α) is bounded and closed.
E 7. Every bounded convex set has content.

6. **Some lemmas on the triangles inscribable in a given set.** The *diameter* of a set α is the least upper bound of the distance PQ for all pairs of points P, Q in α.

We shall represent a set and its content (if existent) by the same symbol. If the content is not known to exist, the upper content will be denoted by the symbol for the set with a bar over it.

A triangle PQR is *inscribed* in a set α if the points P, Q, R are in α.

LEMMA T 1. *In any closed bounded convex set α there may be inscribed a $\triangle PQR$ whose area is at least one-fourth the content of α and whose longest side is equal in length to the diameter of α.*

Take two points P, Q of α whose distance apart is equal to the diameter of α. Let p, q be the lines through P, Q respectively and $\perp$ to the line PQ. Clearly all points of α lie between or on the lines p, q. Let P_1P_2, Q_1Q_2 be the segments which are the projections of α on p and q respectively. Then α lies entirely in the rectangle $P_1P_2Q_2Q_1$. Let R_1, R_2 be points of α on P_1Q_1, P_2Q_2 respectively. Let R be that one of the two points R_1 R_2 which is at the greater distance from PQ. Then

$$\triangle PQR \geqq \frac{1}{2}(\triangle PQR_1+\triangle PQR_2) \geqq \frac{1}{4}\text{rectangle } P_1Q_1Q_2P_2 \geqq \frac{\alpha}{4}.$$

LEMMA T 2. *If P, Q are points of* E (α) *such that PQ equals the diameter of* E (α), *then P, Q are in α.*

For if one of them, say Q, is not in α it is collinear with two points Q', Q'' in α. But then one of the distances PQ', PQ'' would exceed PQ and PQ would not be a diameter of E (α).

LEMMA T 3. *If P, Q are points of α and R_0 is in* E (α), *there is a point R of α such that $\triangle PQR \geqq \triangle PQR_0$.*

If R_0 is in α take $R = R_0$. If R_0 is not in α it is collinear with two points R_0', R_0'' of α, one of which is as far from line PQ as R_0 and may be taken as R.

LEMMA T 4. *In any closed bounded set α may be inscribed a $\triangle$ whose area is at least one-fourth the content of* $E^2(\alpha)$.

There is (by Lemma T 1) a $\triangle PQR_0$ inscribed in $E^2(\alpha)$ such that $\triangle PQR \geqq \frac{1}{4}E^2(\alpha)$, and whose longest side, PQ say, is a diameter of $E^2(\alpha)$. By two applications (at most) of Lemma T 2 it is seen that P, Q are in α. By at most two applications of Lemma T 3 a point R in α may be found such that $\triangle PQR \geqq \triangle PQR_0$.

LEMMA T 5. *In any bounded set α there may be inscribed a $\triangle$ whose area is at least one-fifth the upper content of* $E^2(\alpha)$.

This is proved by applying Lemma T 4 to $E^2(\alpha + \alpha')$.

7. The necessary conditions A. By $\alpha_{\varphi\psi}(I)$ will be denoted the set of all points $(\varphi(t), \psi(t))$ for t in I.

INEQUALITY B. $O_I S^0 \psi \Delta \varphi \geqq \frac{1}{5} E^2 \alpha_{\varphi\psi}(I)$.

First there are (by Lemma T 5) three points $(\varphi(t_1), \psi(t_1))$, $(\varphi(t_2), \psi(t_2))$, $(\varphi(t_3), \psi(t_3))$ which form a triangle having area exceeding one-fifth the content of $E^2 \alpha_{\varphi\psi}(I)$. That is,

$$\left| \frac{1}{2}\{\psi(t_1)+\psi(t_2)\}[\varphi(t_2)-\varphi(t_1)] + \frac{1}{2}\{\psi(t_2)+\psi(t_3)\}[\varphi(t_3)-\varphi(t_2)] \right.$$
$$\left. + \frac{1}{2}\{\psi(t_3)+\psi(t_1)\}[\varphi(t_1)-\varphi(t_3)] \right| \geqq \frac{1}{5} E^2 \alpha_{\varphi\psi}(I).$$

Now take $\pi' = \underline{I} t_1 t_2 t_3 \overline{I}$, $\pi'' = \underline{I} t_1 t_3 \overline{I}$. Then

$$|S^0_{\pi'} \psi \Delta \varphi - S^0_{\pi''} \psi \Delta \varphi| \geqq \frac{1}{5} E^2 \alpha_{\varphi\psi}(I);$$

from which the inequality follows.

The symbol $S_\pi E^2 \Delta \alpha_{\varphi\psi}$ will denote $\sum^{\pi}_{\Delta\pi} E^2 \alpha_{\varphi\psi}(\Delta\pi)$.

THEOREM N 3. *In order that $\int_0^1 \psi d\varphi$ exist in the sense (FS) or (FW) it is necessary that*

$A_F(\varphi\psi)$: $\quad L_F S E^2 \Delta \alpha_{\varphi\psi} = 0;$

in the sense (NS) or (NW) that

$A_N(\varphi\psi)$: $\quad L_N S E^2 \Delta \alpha_{\varphi\psi} = 0.$

COROLLARY. *In order that $\int_0^1 \psi d\varphi$ exist in either sense it is necessary that content $\alpha_{\varphi\psi}(T)$ be zero.*

In § 12 will be given an example to show that this is not sufficient even when $\alpha_{\varphi\psi}(T)$ is a simple (continuous) arc.

8. **Lemmas. The relation** U_0. A function $\varphi(t)$ gives rise to an associated function $\varphi_\pi(t)$ relative to π defined as follows:

$$\varphi_\pi(t) = \varphi(\underline{\Delta\pi}) + \varphi(\underline{\Delta\pi}, \overline{\Delta\pi})(t - \underline{\Delta\pi}) \text{ for } t \text{ in } \Delta\pi \text{ and each } \Delta\pi,$$

where

$$\varphi(t_1, t_2) = \frac{\varphi(t_1) - \varphi(t_2)}{t_1 - t_2}.$$

The following algebraic identity

$$\sum_{i=1}^{n-1}(y_i + y_{i+1})(x_{i+1} - x_i) = (y_1 + y_n)(x_n - x_1) - \sum_{i=2}^{n-1}\begin{vmatrix} x_1 & y_1 & 1 \\ x_i & y_i & 1 \\ x_{i+1} & y_{i+1} & 1 \end{vmatrix}$$

is easily proved by induction. By its aid it is easily shown that

$$S^0_{\pi'}\, \psi_\pi\, d\varphi_\pi = S^0_\pi\, \psi\, \Delta\varphi \qquad (\pi' \mathrm{F}\, \pi),$$

and hence that

$$\int_0^1 \psi_\pi\, d\varphi_\pi = S^0_{\pi'}\, \psi_\pi\, \Delta\varphi_\pi = S^0_\pi\, \psi\Delta\varphi \qquad (\pi' \mathrm{F}\, \pi).$$

$\varphi\, U_0 \psi$ on I if $\varphi(I')^2 + \psi(I')^2 > 0$ for every I' within I such that $O_{I'}\, \varphi + O_{I'}\, \psi > 0$.

$\varphi\, U_{00}\, \psi$ on I if there is a $\pi_0(I)$ such that $\varphi\, U_0\, \psi$ on each $\Delta\pi_0(I)$.

$\varphi\, U\, \psi$ on I if there is a $\pi_0(I)$ such that for every $\pi(I) \mathrm{F}\, \pi_0(I)$ there is a $\pi'(I)$ such that $\pi'(I) \mathrm{F}\, \pi(I)$ and $\varphi_{\pi'}\, U_0\, \psi_{\pi'}$ on each $\Delta\pi_0(I)$.

LEMMA U_0 1. *If* $\varphi_\pi\, U_0\, \psi_\pi$ *on* I *and*

$$\begin{vmatrix} \varphi_\pi(t) & \psi_\pi(t) & 1 \\ \varphi_\pi(\underline{I}) & \psi_\pi(\underline{I}) & 1 \\ \varphi_\pi(\overline{I}) & \psi_\pi(\overline{I}) & 1 \end{vmatrix} \neq 0 \qquad (\underline{I} < t < \overline{I}),$$

then

$$\left| \frac{1}{2}\{\psi_\pi(\underline{I}) + \psi_\pi(\overline{I})\}\, \varphi_\pi(I) - \int_I \psi_\pi\, d\varphi_\pi \right| \leqq \mathrm{E}^2 \alpha_{\varphi_\pi \psi_\pi}(\underline{I}).$$

For if $\pi(I)$ is $t_0 = \underline{I}, t_1, t_2, \cdots, t_{n-1}, t_n = \overline{I}$ and the P_i are the points $(\varphi_\pi(t_i), \psi_n(t_i))$ $(i = 0, \cdots, n)$, the polygon $P_0 P_1 \cdots P_n P_0$ is simple and its area the left-hand side of the inequality while the right-hand side is the area of the smallest convex polygon which contains the polygon $P_0 P_1 \cdots P_n P_0$.

9. Reduction of functions to be in the U_0 relation. Some inequalities. Let φ, ψ be continuous on $I(\subseteq T)$. If $\varphi \,\overline{\ }\, U_0 \psi$ on I, there exists uniquely a sequence $\{G_n\}$ of intervals defined by induction as follows:

(i) G_1 is such that

$$(1)\ G \subseteq I; \qquad (2)\ O_{G_1}\varphi + O_{G_1}\psi > 0; \qquad (3)\ \varphi(G_1)^2 + \psi(G_1) = 0;$$
$$(4)\ \varphi(G)^2 + \psi(G)^2 > 0$$

for every $G \subseteq I$ such that $O_G\varphi + O_G\psi > 0$ and such that $\bar{G} - \underline{G} > \bar{G}_1 - \underline{G}_1$ or $\bar{G} - \underline{G} = \bar{G}_1 - \underline{G}_1$ and $\underline{G} < \underline{G}_1$;

(ii) $G_1, \cdots, G_n$ having been defined, G_{n+1} is such that

$$(0)\ G_{n+1} \cdot (G_1 + \cdots + G_n) = 0; \quad (1)\ G_{n+1} \subseteq I; \quad (2)\ O_{G_{n+1}}\varphi + O_{G_{n+1}}\psi > 0;$$
$$(3)\ \varphi(G_{n+1})^2 + \psi(G_{n+1})^2 = 0; \qquad (4)\ \varphi(G)^2 + \psi(G)^2 > 0$$

for every $G \subseteq I$ such that $G \cdot (G_1 + \cdots + G_n) = 0$, $O_G\varphi + O_G\psi > 0$ and such that $\bar{G} - \underline{G} > \bar{G}_{n+1} - \underline{G}_{n+1}$ or $\bar{G} - \underline{G} = \bar{G}_{n+1} - \underline{G}_{n+1}$ and $\underline{G} < \underline{G}_{n+1}$.

It is clear that $G_i \cdot G_j = 0$ for every $i \neq j$, and, therefore, if the sequence is infinite

$$\mathop{\mathrm{L}}_{n}(\bar{G}_n - \underline{G}_n) = 0.$$

We may now define two functions $\varphi_{\{\psi I\}}(t)$, $\psi_{\{\varphi I\}}(t)$ over T as follows:

$$\begin{aligned}
\varphi_{\{\psi I\}}(t) &= \varphi(t) \text{ if } t \text{ is in } T - \sum_n G_n; \\
&= \varphi(\underline{G}_n) = \varphi(\bar{G}_n) \text{ if } t \text{ is in } G_n \qquad (n); \\
\psi_{\{\varphi I\}}(t) &= \psi(t) \text{ if } t \text{ is in } T - \sum_n G_n; \\
&= \psi(\underline{G}_n) = \psi(\bar{G}_n) \text{ if } t \text{ is in } G_n \qquad (n).
\end{aligned}$$

LEMMA U_0 2. *$\varphi_{\{\psi I\}}\, U_0\, \psi_{\{\varphi I\}}$ on I.*

Take G so that $O_G\varphi_{\{\psi I\}} + O_G\psi_{\{\varphi I\}} > 0$. We are to prove that $\varphi_{\{\psi I\}}(G)^2 + \psi_{\{\varphi I\}}(G)^2 > 0$. We note first that the condition on G implies that G is not entirely in any G_n.

Suppose first that $G \cdot (\sum_n G_n) = 0$. Take n so that $\bar{G} - \underline{G} > \bar{G}_n - \underline{G}_n$. This together with $G \subseteq I$, $G \cdot (G_1 + \cdots + G_{n-1}) = 0$ and

$$O_G\varphi + O_G\psi = O_G\varphi_{\{\psi I\}} + O_G\psi_{\{\varphi I\}} > 0$$

shows that $\varphi(G)^2 + \psi(G)^2 > 0$. But $\varphi_{\{\psi I\}}(G)^2 + \psi_{\{\varphi I\}}(G)^2 = \varphi(G)^2 + \psi(G)^2$. Hence $\varphi_{\{\psi I\}}(G)^2 + \psi_{\{\varphi I\}}(G)^2 > 0$.

Suppose $G \cdot (\sum_n G_n) > 0$. Let n be the smallest integer for which $G_n \cdot G > 0$, and set $G_0 = G_n + G$. Then $\overline{G}_0 - \underline{G}_0 > \overline{G}_n - \underline{G}_n$ (since G does not lie entirely in G_n) and this together with $G_0 \subseteq I$, $G_0 \cdot (G_1 + \cdots + G_{n-1}) = 0$, $O_{G_0}\varphi + O_{G_0}\psi \geqq O_G \varphi_{\{\psi I\}} + O_{G_{\{\varphi I\}}} > 0$ shows that $\varphi(G_0)^2 + \psi(G_0)^2 > 0$. But $\varphi_{\{\psi I\}}(G)^2 + \psi_{\{\varphi I\}}(G)^2 = \varphi(G)^2 + \psi(G)^2$. Hence $\varphi_{\{\psi I\}}(G)^2 + \psi_{\{\varphi I\}}(G)^2 > 0$.

The lemma is thus completely proved.

We now define $\varphi_{\{\psi\pi\}}$, $\psi_{\{\varphi\pi\}}$ as follows:

$$\varphi_{\{\psi\pi\}}(t) = \varphi_{\{\psi\Delta\pi\}}(t) \text{ if } t \text{ is in } \Delta\pi \quad (\Delta\pi);$$

$$\psi_{\{\varphi\pi\}}(t) = \psi_{\{\varphi\Delta\pi\}}(t) \text{ if } t \text{ is in } \Delta\pi \quad (\Delta\pi).$$

It is easily seen that there is no conflict of definition at the division points of π.

We now define integration processes $\int_{\{I\}}$, $\int_{\{\pi\}}$, $\left|\int\right|$, $a\int + b\int_{\{\pi\}}$, etc., thus:

$$\int_{\{I\}} \psi\, d\varphi \equiv \int \psi_{\{\varphi I\}}\, d\varphi_{\{\psi I\}},$$

$$\int_{\{\pi\}} \psi\, d\varphi \equiv \int \psi_{\{\varphi\pi\}}\, d\varphi_{\{\psi\pi\}},$$

$$\left|\int\right| \psi\, d\varphi \equiv \left|\int \psi\, d\varphi\right|,$$

$$\left(a\int + b\int_{\{\pi\}}\right) \psi\, d\varphi \equiv a\int \psi\, d\varphi + b\int_{\{\pi\}} \psi\, d\varphi, \text{ etc.}$$

If $f(\pi)$ has a meaning for every partition π of T or of a sub-interval I, we shall denote by $\overline{\mathrm{B}}_I f$ the least upper bound of $f(\pi)$ for all partitions of I and by $\overline{\mathrm{B}}_\pi f$ the sum $\sum_{\Delta\pi}^{\pi} \overline{\mathrm{B}}_{\Delta\pi} f$.

Inequality $\{I\}$.

$$\left| \frac{1}{2}\{\psi_{\pi(I)}(\underline{I}) + \psi_{\pi(I)}(\overline{I})\}\, \varphi_{\pi(I)}(I) - \int_{\{I\}_I} \psi_{\pi(I)}\, d\varphi_{\pi(I)} \right| \leq 2\overline{\mathrm{B}}_I S \mathrm{E}^2 \Delta \alpha_{\varphi\psi}.$$

INEQUALITY $\{\pi\}$.

$$\left| \int \psi_\pi \, d\varphi_\pi - \int_{\{\pi\}} \psi_{\pi'} \, d\varphi_{\pi'} \right| \leqq 2\,\overline{\mathrm{B}}_\pi \, S \mathrm{E}^2 \Delta \, \alpha_{\varphi\psi} \quad (\pi' \, \mathrm{F} \, \pi).$$

It is clear that Inequality $\{\pi\}$ follows from Inequality $\{I\}$, which we now prove.

Let $t_1 < t_2 < \cdots < t_{n-1}$ be the values for which

$$\begin{vmatrix} \varphi_{\pi\{\psi_\pi I\}}(t) & \psi_{\pi\{\varphi_\pi I\}}(t) & 1 \\ \varphi_\pi(\underline{I}) & \psi_\pi(\underline{I}) & 1 \\ \varphi_\pi(\overline{I}) & \psi_\pi(\overline{I}) & 1 \end{vmatrix} = 0,$$

and π_0 the partition $\underline{I} = t_0, t_1, \cdots, t_{n-1}, t_n = \overline{I}$.

Now by the algebraic identity of § 8

$$\frac{1}{2}\{\psi_\pi(\underline{I}) + \psi_\pi(\overline{I})\}\, \varphi_\pi(I) - \int_{\{I\}I} \psi_\pi \, d\varphi_\pi$$

$$= \sum_{i=1}^{n} \left[\frac{1}{2}\{\psi_\pi(t_{i-1}) + \psi_\pi(t_i)\}\{\varphi_\pi(t_i) - \varphi_\pi(t_{i-1})\} - \int_{t_{i-1}}^{t_i} \psi_{\pi\{\varphi_\pi I\}} \, d\varphi_{\pi\{\psi_\pi I\}} \right].$$

Hence by Lemma $\mathrm{U}_0 1$

$$\left| \frac{1}{2}\{\psi_\pi(\underline{I}) + \psi_\pi(\overline{I})\}\, \varphi_\pi(I) - \int_{\{I\}I} \psi_\pi \, d\varphi_\pi \right| \leqq \sum_{\Delta\pi_0}^{\pi_0} \mathrm{E}^2 \alpha_{\psi_\pi \varphi_\pi}(\Delta \pi_0).$$

The partition π_0 is such that between any two of its consecutive division points t_{i-1}, t_i there is a division point u_i of π. Now let π', π'' be the partitions

$$\pi' : \quad \underline{I}, u_1, u_3, u_5, \cdots, \overline{I},$$

$$\pi'' : \quad \underline{I}, u_2, u_4, u_6, \cdots, \overline{I}.$$

Then every cell $\Delta \pi_0$ is in a cell $\Delta \pi'$ or in a cell $\Delta \pi''$.

Therefore

$$\begin{aligned} \sum_{\Delta\pi_0}^{\pi_0} \mathrm{E}^2 \alpha_{\varphi_\pi \psi_\pi}(\Delta \pi_0) &\leqq \sum_{\Delta\pi'}^{\pi'} \mathrm{E}^2 \alpha_{\varphi_\pi \psi_\pi}(\Delta \pi') + \sum_{\Delta\pi''}^{\pi''} \mathrm{E}^2 \alpha_{\varphi_\pi \psi_\pi}(\Delta \pi'') \\ &\leqq \sum_{\Delta\pi'}^{\pi'} \mathrm{E}^2 \alpha_{\varphi\psi}(\Delta \pi') + \sum_{\Delta\pi''}^{\pi''} \mathrm{E}^2 \alpha_{\varphi\psi}(\Delta \pi'') \\ &\leqq 2\,\overline{\mathrm{B}}_I \, S \mathrm{E}^2 \alpha_{\varphi\psi}, \end{aligned}$$

which establishes Inequality $\{I\}$.

10. The necessary condititions J. The identity

$$\int \psi_{\pi'}\, d\varphi_{\pi'} - \int \psi_{\pi''}\, d\varphi_{\pi''} = \int \psi_{\pi'}\, d\varphi_{\pi'} - \int_{\{\pi'\}} \psi_{\pi'''}\, d\varphi_{\pi'''}$$
$$+ \int_{\{\pi'\}} \psi_{\pi'''}\, d\varphi_{\pi'''} - \int \psi_{\pi'''}\, d\varphi_{\pi'''}$$
$$+ \int \psi_{\pi'''}\, d\varphi_{\pi'''} - \int_{\{\pi''\}} \psi_{\pi'''}\, d\varphi_{\pi'''}$$
$$+ \int_{\{\pi''\}} \psi_{\pi'''}\, d\varphi_{\pi'''} - \int \psi_{\pi''}\, d\varphi_{\pi''}$$

implies the inequalities J 1, J 2.

Inequality J 1.

$$\left| \int \psi_{\pi'}\, d\varphi_{\pi'} - \int \psi_{\pi''} - d\varphi_{\pi''} \right|$$
$$\leqq 2[\bar{\mathrm{B}}_{\pi'}\, S\mathrm{E}^2 \Delta\, \alpha_{\varphi\psi} + \bar{\mathrm{B}}_{\pi''}\, S\mathrm{E}^2 \Delta\, \alpha_{\varphi\psi}]$$
$$+ \left[\left| \int_{\{\pi'\}} \psi_{\pi'''}\, d\varphi_{\pi'''} - \int \psi_{\pi'''}\, d\varphi_{\pi'''} \right| \right.$$
$$\left. + \left| \int_{\{\pi''\}} \psi_{\pi'''}\, d\varphi_{\pi'''} - \int \psi_{\pi'''}\, d\varphi_{\pi'''} \right| \right] \qquad (\pi''' \mathrm{F}\, \pi',\ \pi''' \mathrm{F}\, \pi'').$$

Inequality J 2.

$$\left| \int_{\{\pi'\}} \psi_{\pi'''}\, d\varphi_{\pi'''} - \int \psi_{\pi'''}\, d\varphi_{\pi'''} \right| + \left| \int_{\{\pi''\}} \psi_{\pi'''}\, d\varphi_{\pi'''} - \int \psi_{\pi'''}\, d\varphi_{\pi'''} \right|$$
$$\leqq \left| \int \psi_{\pi'}\, d\varphi_{\pi'} - \int \psi_{\pi''}\, d\varphi_{\pi''} \right|$$
$$+ 2[\bar{\mathrm{B}}_{\pi'}\, S\mathrm{E}^2 \Delta\, \alpha_{\varphi\psi} + \bar{\mathrm{B}}_{\pi''}\, S\mathrm{E}^2\, \alpha\, \Delta_{\varphi\psi}] \qquad (\pi''' \mathrm{F}\, \pi',\ \pi''' \mathrm{F}\, \pi'').$$

Theorem N 4. *The conditions*

$$J_{\mathrm{F}}(\varphi\, \psi):\ \underset{\pi'\pi''}{\mathrm{L_F}}\ \underset{\pi'''}{L_{\mathrm{F}}} \left[\left| \int_{\{\pi'\}} - \int \right| + \left| \int_{\{\pi''\}} - \int \right| \right] \psi_{\pi'''}\, d\varphi_{\pi'''} = 0,$$

$$J_{\mathrm{N}}(\varphi\, \psi):\ \underset{\pi'\pi''}{\mathrm{L_N}}\ \underset{\pi'''}{L_{\mathrm{N}}} \left[\left| \int_{\{\pi'\}} - \int \right| + \left| \int_{\{\pi''\}} - \int \right| \right] \psi_{\pi'''}\, d\varphi_{\pi'''} = 0$$

are respectively necessary for the existence of $\int_0^1 \psi\, d\varphi$ *in the senses* (*FW*), (*NW*) *and therefore respectively necessary for its existences in the senses* (*FS*), (*NS*).

This theorem follows from inequality J 2.

11. **The general existence theorem.** By $S_\pi(\Delta O\psi)|\Delta\varphi|$ will be denoted $\sum_{\Delta\pi}^{\pi}(O_{\Delta\pi}\psi)|\varphi(\Delta\pi)|$, and by $\overline{\mathrm{B}}_I S(\Delta O\psi)|\Delta\varphi|$ will be denoted the least upper bound of $S_\pi(\Delta O\psi)|\Delta\varphi|$ for partitions π of I; and by $\overline{\mathrm{B}}_\pi S(\Delta O\psi)|\Delta\varphi|$ will be denoted $\sum_{\Delta\pi}^{\pi}\overline{\mathrm{B}}_{\Delta\pi}S(\Delta O\psi)|\Delta\varphi|$.

INEQUALITY S_0. $\mathrm{E}^2\alpha_{\varphi\psi}(I)\leqq 5\overline{\mathrm{B}}_I S(\Delta O\psi)|\Delta\varphi|$.

For let t_1, t_2, t_3 be taken in I so that

$$\Big|\frac{1}{2}\{\psi(t_1)+\psi(t_2)\}[\varphi(t_2)-\varphi(t_1)]+\frac{1}{2}\{\psi(t_2)+\psi(t_3)\}[\varphi(t_3)-\varphi(t_2)]$$

$$+\frac{1}{2}\{\psi(t_3)+\psi(t_1)\}[\varphi(t_1)-\varphi(t_3)]\Big|\leqq\frac{1}{5}\mathrm{E}^2\alpha_{\varphi\psi}(I).$$

Then

$$|\psi(t_1)-\psi(t_2)|\,|\varphi(t_1)-\varphi(t_2)|+|\psi(t_2)-\psi(t_3)|\,|\varphi(t_2)-\varphi(t_3)|$$

$$+|\psi(t_3)-\psi(t_1)|\,|\varphi(t_3)-\varphi(t_1)|\geqq\frac{2}{5}\mathrm{E}^2\alpha_{\varphi\psi}(I)$$

by the algebraic inequality

$$|(x_1-x_2)(y_1+y_2)+(x_2-x_3)(y_2+y_3)+(x_3-x_1)(y_3+y_1)|$$

$$\leqq|x_1-x_2|\,|y_1-y_2|+|x_2-x_3|\,|y_2-y_3|+|x_3-x_1|\,|y_3-y_1|,$$

which is easily proved.

Now let $\pi'=\underline{I}, t_1, t_2, t_3, \overline{I}$ and $\pi''=\underline{I}, t_1, t_3, \overline{I}$. Then

$$S_{\pi'}(\Delta O\psi)|\Delta\varphi|+S_{\pi''}(\Delta O\psi)|\Delta\varphi|\geqq\frac{2}{5}\mathrm{E}^2\alpha_{\varphi\psi}(I);$$

from which the inequality follows.

INEQUALITY S. $S_\pi\mathrm{E}^2\Delta\alpha_{\varphi\psi}\leqq 5\overline{\mathrm{B}}_\pi S(\Delta O\psi)|\Delta\varphi|$.

Let us introduce the conditions

$$O_F:\qquad \mathrm{L}_F S(\Delta O\psi)|\Delta\varphi|=0.$$

$$O_N:\qquad \mathrm{L}_N S(\Delta O\psi)|\Delta\varphi|=0.$$

We can now state the

EXISTENCE THEOREM. *The four pairs of conditions*

$$\begin{array}{ll} A_F(\varphi\psi), & J_F(\varphi\psi);\\ A_N(\varphi\psi), & J_N(\varphi\psi);\\ J_F(\varphi\psi), & O_F(\varphi\psi);\\ J_N(\varphi\psi), & O_N(\varphi\psi)\end{array}$$

are respectively necessary and sufficient for the existence of $\int_0^1 \psi\, d\varphi$ *in the senses* (FW), (NW), (FS), (NS).

The necessity of the various conditions has already been proved. The sufficiency of the first two pairs follows from inequality J_1 on operating on both sides of that inequality by $\underset{\pi'\pi''}{\mathrm{L_F}}\, \underset{\pi'''}{L_F}$, $\underset{\pi'\pi''}{\mathrm{L_N}}\, \underset{\pi'''}{L_N}$ respectively. The sufficiency of the last two pairs then follows from inequality S_0.

LEMMA U 1. *If* $\varphi\, \mathrm{U}\, \psi$ *on* T, *then* $J_F(\varphi\,\psi)$ *and* $J_N(\varphi\,\psi)$.

LEMMA U 2. *If* $\alpha_{\varphi\psi}(T)$ *is a continuous arc with at most a finite number of multiple points, then* $\varphi\, \mathrm{U}\, \psi$ *on* T.

This is easily shown by slightly modifying a proof of de la Vallée Poussin (see Pierpont, *Theory of Functions of Real Variables*, vol. II, p. 597).

LEMMA U 3. *If* φ *is monotone on* T, *then* $\varphi\, \mathrm{U}\, \psi$.

COROLLARY 1. *The four conditions* A_F, A_N, O_F, O_N *are respectively necessary for the existence of* $\int_0^1 \psi\, d\varphi$ *in the senses* (FW), (NW), (FS), (NS); *and are respectively sufficient if* $\varphi\, \mathrm{U}\, \psi$ *on* T, *in particular, if* $\alpha_{\varphi\psi}(T)$ *is a continuous curve with a finite number of multiple points, or if* φ *is a monotone function on* T.

Suppose φ is of limited variation on T, that is, that $\int_0^1 |d\varphi|$ exists. Then $\int_I |d\varphi|$ exists for every I in T and will be denoted by $\mathrm{V}_\varphi(I)$. We define two functions $\varphi_1(t)$, $\varphi_2(t)$ on T by the equations

$$\varphi_1(t) = \frac{1}{2}\left\{\int_0^t |d\varphi| + [\varphi(t) - \varphi(0)]\right\} + \varphi(0),$$

$$\varphi_2(t) = \frac{1}{2}\left\{\int_0^t |d\varphi| - [\varphi(t) - \varphi(0)]\right\},$$

so that

$$\varphi(t) = \varphi_1(t) - \varphi_2(t)$$

and

$$\Delta\varphi_1 = \frac{1}{2}\{\Delta \mathrm{V}_\varphi + \Delta\varphi\},$$

$$\Delta\varphi_2 = \frac{1}{2}\{\Delta \mathrm{V}_\varphi - \Delta\varphi\}.$$

Hence φ_1, φ_2 are monotonic increasing functions in view of the obvious inequality

$$|\Delta\varphi| \leq \Delta \mathrm{V}_\varphi.$$

We have, if $O_T\,\psi$ is finite,

$$\mathrm{L_F}\, S(\Delta O\psi)\{|\Delta\varphi| - \Delta \mathrm{V}_\varphi\} = \mathrm{L_N}\, S(\Delta O\psi)\{|\Delta\varphi| - \Delta \mathrm{V}_\varphi\} = 0;$$

from which it follows that the conditions

$$V_F(\varphi\psi): \qquad \mathrm{L}_F\, S(\Delta O\psi)\Delta \mathrm{V}_\varphi = 0,$$
$$V_N(\varphi\psi): \qquad \mathrm{L}_N\, S(\Delta O\psi)\Delta \mathrm{V}_\varphi = 0$$

are respectively equivalent to the conditions $O_F(\varphi\psi)$, $O_N(\varphi\psi)$, so that $V_F(\varphi\psi)$, $V_N(\varphi\psi)$ are necessary conditions for the existence of $\int_0^1 \psi\, d\varphi$ in the senses (FS), (NS) respectively. Moreover the equations

$$\Delta \mathrm{V}_\varphi = \Delta\varphi_1 + \Delta\varphi_2,$$
$$\Delta\varphi = \Delta\varphi_1 - \Delta\varphi_2$$

show that the conditions $V_F(\varphi\psi)$, $V_N(\varphi\psi)$ imply respectively the pair of conditions $V_F(\varphi_1\psi)$, $V_F(\varphi_2\psi)$ and $V_N(\varphi_1\psi)$, $V_N(\varphi_2\psi)$. Hence the conditions $V_F(\varphi\psi)$, $V_N(\varphi\psi)$ imply the existence of $\int_0^1 (\psi\, d\varphi)$ in the respective senses (FS), (NS) with the value

$$\int_0^1 \psi\, d\varphi = \int_0^1 \psi\, d\varphi_1 - \int_0^1 \psi\, d\varphi_2.$$

We summarize these well known results as

COROLLARY 2. *If φ is of limited variation and $O_T\psi$ is finite, either of the conditions V_F $(\varphi\psi)$, O_F $(\varphi\psi)$ is necessary and sufficient for the existence of $\int_0^1 \psi\, d\varphi$ in the sense (FS) and either of the conditions V_N $(\varphi\psi)$, $O_N(\varphi\psi)$ is necessary and sufficient for the existence of $\int_0^1 \psi\, d\varphi$ in the sense (NS).*

12. **A squarable crinkly curve whose associated Stieltjes integral fails to exist.** If $P_1, \cdots, P_n$ are any n points in a plane in which a system of rectangular coördinates has been established, let $(P_1, \cdots, P_n)$ be defined by the equation

$$(P_1, \cdots, P_n) = \sum_{i=1}^{n} \frac{1}{2}\{\text{ord } P_i + \text{ord } P_{i+1}\}\ [\text{abs } P_{i+1} - \text{abs } P_i].$$

Now let S denote a square of which two sides are parallel to and above the x-axis. Let us agree to denote the area of any geometric figure by the same letter as the figure so that S will also denote the area of the square S. Let AB represent one diagonal of S. Finally let f be any positive integer.

Take a positive integer p. Divide S into p^2 equal squares. Then divide each of these squares into p^2 equal squares, and so on. In this way we

secure an infinite sequence of divisions of S into p^2, p^4, $p^6, \cdots$ equal squares. The vertices of these squares form a set $[X]$ everywhere dense in S. The number m such that p^{2m} is the smallest number (a power of p^2) of equal squares into which S may be divided so that X appears as a vertex is called the order of X.

Denote by M that one of the vertices of S other than A and S for which it is true that $(ABMA)$ is positive.

Let $A_1B_1, \cdots, A_rB_r$ be r sub-segments of the segment AB with end points in $[X]$ and such that A_{i+1} is between $A_i (i = 1, \cdots, r)$ and any B_j $(j = 1, \cdots, r)$, and B_{i+1} is between B_i and any A_j and such moreover that

$$\overline{A_1B_1}^2 + \cdots + \overline{A_rB_r}^2 > 2f \cdot \overline{AB}^2.$$

Now let $M_1, \cdots, M_r$ be r points all on the same side of AB as M and such that $A_iM_iB_i$ is a right angle $(i = 1, \cdots, r)$. Then the points M_i are in $[X]$, the quantities $(A_iB_iM_iA_i)$ are positive, and the broken lines $A_iM_iB_i$ do not have any points in common.

Next take any point A_0 of the set $[X]$ which is within the segment AA_1 and then choose $N_1, \cdots, N_r$ all on the opposite side of AB from M so that the angles $A_{i-1}N_iB_i$ shall all be right angles. Then $N_1, \cdots, N_r$ are in $[X]$, the quantities $(A_{i-1}N_iB_iA_{i-1})$ are positive, and the broken lines $A_{i-1}N_iB_i$, $B_jM_jA_j$ have no points (other than end points) in common.

Thus the points

$$AA_0, \ N_1B_1M_1A_1, \cdots, \ N_iB_iM_iA_i, \cdots, \ N_rB_rM_rA_r$$

taken in order form the vertices in $[X]$ of a simple broken line which joins A to A_r, and consists of segments each of which, except AA_0, is parallel to a coördinate axis. It is clearly possible to join A_r to B by a broken line of the same character which does not have any point other than A_r in common with this broken line. Let $A_rQ_1 \cdots Q_sB$ denote such a broken line. Then

$$AA_0N_1B_1M_1A_1 \cdots N_iB_iM_iA_i \cdots N_rB_rM_rA_rQ_1 \cdots Q_sB$$

taken in order are the vertices of a simple broken line $A\lambda_0B$ which consists, aside from the segments AA_0 and Q_sB, entirely of segments parallel to the axes of coördinates.

The points

$$AB_1M_1A_1 \cdots B_iM_iA_i \cdots B_rM_rA_rB$$

taken in order form the vertices of a broken line $A\lambda' B$ which is inscribed in $A\lambda_0 B$.

If we note that for any three collinear points PQR it is true that $(PQR) = (PQ) + (QR)$, then we see that

$$\begin{aligned}(A\lambda' B) &= (AA_1) + (A_1 B_1 M_1 A_1) + (A_1 A_2) + (A_2 B_2 M_2 A_2) + \cdots \\ &\quad + (A_{i-1} A_i) + (A_i B_i M_i A_i) + \cdots \\ &\quad + (A_{r-1} A_r) + (A_r B_r M_r A_r) + (A_r B) \\ &= (AB) + \sum_{i=1}^{r} (A_i B_i M_i A_i) > (AB) + fS,\end{aligned}$$

so that

$$(A\lambda' B) - (AB) > fS.$$

The vertices of $A\lambda_0 B$ are all in $[X]$ and hence there is a finite least upper bound k_0 for their orders. Let l be the length of $A\lambda_0 B$. Take $k = k_0 + 2$ so that

$$l \frac{\sqrt{S}}{p^k} < \frac{S}{2} \quad \text{and} \quad < (A\lambda' B) - (AB) - fS.$$

Now suppose S divided into p^{2k} equal squares. Shade all of these squares which have a side in common with a segment of $A\lambda_0 B$ and which lie on the same side of $A\lambda_0 B$ as M does. Let us now suppose that p is even. Then there is an even number of shaded squares against each segment of $A\lambda_0 B$ except AA_0 and $Q_s B$. With this exception, then, it is possible to replace each segment of $A\lambda_0 B$ by a broken line joining the end points of that segment and made up by taking one diagonal from each of the shaded squares that abut thereon. After this has been done and all segments deleted which enter twice in opposite senses, there is obtained a simple broken line $A\lambda B$ all of whose segments, except AA_0 and $Q_s B$, are diagonals of shaded squares. This exception can be removed by shading also all the squares of our division which have an interior point in common with AA_0 or $Q_s B$ and then regarding all vertices of these new shaded squares which lie on AA_0 and on BQ_s as vertices of $A\lambda B$. It is clear that $A\lambda' B$ is inscribed in $A\lambda B$.

We have thus shown how to replace the diagonal AB of S by a simple broken line $A\lambda B$ subject to the following conditions:

(A) the broken line $A\lambda B$ consists of diagonals of certain shaded squares of a division of S into equal squares;

(B) the squares of which the segments of $A\lambda B$ are diagonals have total area less than $S/2$;

(C) a broken line $A\lambda' B$ can be inscribed in $A\lambda B$ for which

$$(A\lambda' B)-(AB) > fS.$$

We can now construct our crinkly curve.

For the sake of being definite, let S be the unit square $0 \leqq x \leqq 1$, $0 \leqq y \leqq 1$ and A, B be the points (0,0), (1,1), respectively. Then on setting $\lambda_0 = AB$, $S_0 = S$, there is for every sequence $f_1, f_2, \cdots$ of positive numbers a sequence $A\lambda_0 B$, $A\lambda_1 B$, $A\lambda_2 B, \cdots$ of simple broken lines joining A to B and subject to the following conditions:

(A) every segment of $A\lambda_n B$ is a diagonal of a square of a certain division of S into equal squares;

(B) the total area S_{n+1} of the squares whose diagonals are the segments of $A\lambda_{n+1} B$ is less than half the total area S_n of the squares S_n whose diagonals are the segments of $A\lambda_n B$;

(C) every vertex of $A\lambda_n B$ is a vertex of $A\lambda_{n+1} B$;

(D) in $A\lambda_{n+1} B$ may be inscribed a broken line $A\lambda'_{n+1} B$ such that

$$(A\lambda'_{n+1} B)-(A\lambda_n B) > f_{n+1} \cdot S_n.$$

To obtain such a sequence we have only to apply the process above, first to the diagonal of the square S, second to each segment of the broken line $A\lambda_1 B$ so obtained, and so on, at each step taking the proper value for f.

Let

$$A\lambda_n B: \quad x = \varphi_n(t), \quad y = \psi_n(t) \qquad (0 \leqq t \leqq 1)$$

denote a one-to-one representation of $A\lambda_n B$ on to $T = (0, 1)$ such that two equal sub-segments of $A\lambda_n B$ always correspond to two equal subintervals of T. Then φ_n, ψ_n converge to two continuous functions φ, ψ, respectively, since φ_n, ψ_n are continuous and the convergence is uniform.

The arc

$$\Gamma: \quad x = \varphi(t), \quad y = \psi(t) \qquad (0 \leqq t \leqq 1)$$

is a simple continuous arc joining A to B.

The broken lines $A\lambda_n B$ (and hence the broken lines $A\lambda'_n B$) are inscribed in Γ. Moreover a vertex of $A\lambda_n B$ is given by the same value of the parameter t in the equations of $A\lambda_n B$ as in the equations of Γ.

The arc Γ is *squarable*. For it lies entirely in the squares S_n and these have, by (B) above, total area less than $1/2^n$.

Now suppose the numbers $f_1 S_0$, $f_2 S_1$, $f_3 S_2$, $\cdots$ are all bounded from

zero, say all greater than e_0. This is possible since the choice of the f's is absolutely arbitrary. Then

$$(1) \qquad (A\,\lambda'_{n+1}\,B) - (A\,\lambda_n\,B) > e_0 \qquad (n = 0, 1, 2, \cdots).$$

The Stieltjes integral $\int_0^1 \psi(t)\,d\varphi(t)$ *does not exist in either of the four senses.* For first the vertices of $A\,\lambda_n\,B$ and $A\,\lambda'_{n+1}\,B$ correspond to two divisions of T of norm in each case certainly less than $1/p^n$. Moreover $(A\,\lambda'_{n+1}\,B)$, $(A\,\lambda_n\,B)$ represent two sums of the form

$$S^0\,\psi\,\Delta\,\varphi$$

corresponding to those divisions. These facts together with the inequality (1) above show the required non-existence.

13. **On the independence of the sufficient conditions.** In this section we find functions φ, ψ for which $A_F(\varphi, \psi)$, $A_N(\varphi, \psi)$, $O_F(\varphi, \psi)$, $O_N(\varphi, \psi)$ but not $J_N(\varphi, \psi)$ or $J_F(\varphi, \psi)$.

Let a square S be divided into $(2\,p+1)^2$ equal squares. Let AB be opposite vertices and represent the remaining vertices by MN in such a way that $(AMBA) > 0$. The diagonal AB is divided by the network of $(2\,p+1)^2$ equal squares into $2\,p+1$ equal segments whose end points we will denote in order by

$$A\,A_1\,A_2 \cdots A_p\,B_p\,B_{p-1} \cdots B_1\,B.$$

Now take $M_1 \cdots M_p N_1 \cdots N_p$ so that $A_i M_i B_i N_i$ is a square S_i $(i = 1, \cdots, p)$ and M_i, N_i are on the same sides of AB as M, N respectively. Set $A_0 = A$, $B_0 = B$, $S_0 = S$. Denote the squares whose diagonals are respectively $A_i A_{i+1}$, $B_i B_{i+1}$, $A_p B_p$ $(i = 0, \cdots, p-1)$ by σ'_i, σ''_i, σ'_p or σ''_p.

There is a simple broken line $A_i\,\lambda'_i\,B_i$ which consists of diagonals of squares (of our network) which lie in $S_{i-1} - (S_i + \sigma'_{i-1} + \sigma''_{i-1})$ on the same side of AB as M and a simple broken line $B_i\,\lambda''_i\,A_i$ consisting of diagonals of squares which lie in $S_{i-1} - (S_i + \sigma'_{i-1} + \sigma''_{i-1})$ on the same side of AB as N.

Now let $A\,\lambda\,B$ be the broken line

$$A\,A_1\,\lambda'_1\,B_1\,\lambda''_1\,A_1\,A_2\,\lambda'_2\,B_2\,\lambda''_2\,A_2 \cdots A_p\,\lambda'_p\,B_p\,\lambda''_p\,A_p\,B_p\,B_{p-1} \cdots B_1\,B.$$

Then the segments of $A\,\lambda\,B$ are diagonals of the $(2p+1)^2$ equal squares into which S has been divided. Moreover

$$
\begin{aligned}
(A\lambda B &= (A A_1) + (A_1 \lambda_1' B_1 \lambda_1'' A_1) + (A_1 A_2) + (A_2 \lambda_2' B_2 \lambda_2'' A_2) + \cdots \\
&\qquad + A_p \lambda_p' B_p \lambda_p'' A_p) + (A_p B_p B_{p-1} \cdots B_1 B) \\
&= (AB) + \sum_{i=1}^{p} (A_i \lambda_i' B_i \lambda_i'' A_i).
\end{aligned}
$$

But

$$
\begin{aligned}
(A_p \lambda_p' B_p \lambda_p'' A_p) &= \frac{2^2}{(2p+1)^2} S, \\
(A_{p-1} \lambda_{p-1}' B_{p-1} \lambda_{p-1}'' A_{p-1}) &= \frac{4^2}{(2p+1)^2} S, \\
\cdots\cdots & \cdots\cdots \\
(A_1 \lambda_1' B_1 \lambda_1'' A_1) &= \frac{(2p)^2}{(2p+1)^2} S.
\end{aligned}
$$

Hence

$$
(A\lambda B) - (AB) = \frac{1}{(2p+1)^2} \{2^2 + 4^2 + \cdots + (2p)^2\} S,
$$

or

$$
(A\lambda B) - (AB) = \frac{2p\,(p+1)}{3\,(2p+1)} S.
$$

Let S be the unit square $0 \leqq x \leqq 1$, $0 \leqq y \leqq 1$ and A, B the points $(0\,0)$, $(1\,1)$ respectively. Let $\{p_n\}$ be any infinite sequence of integers. Set $q_n = 2\,(2p_n + 1)(p_n + 1)$, $m_n = q_1 \cdots q_n$, $r_n = 2p_n + 1$ for $n = 1, 2, \cdots$.

Set $P_{00} = A$, $P_{0m_0} = P_{01} = B$, and let M_{01} be the mid-point of $P_{00}P_{01}$.

There is a broken line $P_{10} \cdots P_{1m_1}$ such that

(1_1) $P_{10} = A$, $P_{1m_1} = B$;

(2_1) the points $P_{10}, \cdots, P_{1r_1-1}$ divide $P_{10}P_{1r_1}$ or $P_{00}M_{01}$ into r_1 equal parts;

(3_1) the points $P_{1r_1}, \cdots, P_{1q_1}$ $(= P_{1m_1})$ are the vertices of a broken line which consists of diagonals of the r_1^2 equal squares into which square $M_{01}P_{01}$ may be divided and which is such that

$$
(P_{1r_1} \cdots P_{1q_1}) - (M_{01}P_{01}) = \frac{2p_1(p_1+1)}{3\,(2p_1+1)} \cdot \mathrm{sq.}\, M_{01}P_{01}.
$$

On continuing in this way we may construct a sequence of broken lines $P_{n0} \cdots P_{nm_n}$ $(n = 1, 2, \cdots)$ such that

(1_n) $$P_{n,kq_n} = P_{n-1,k} \qquad (k = 0, \cdots, m_{n-1});$$

$$P_{n,kq_n+r_n} = M_{n-1,k+1},$$

where $M_{n-1,k+1}$ is the mid-point of $P_{n-1,k}P_{n-1,k+1}$;

(2_n) the points $P_{n,kq_n+1}, \cdots, P_{n,kq_n+r_n-1}$ divide $P_{n,kq_n} P_{n,kq_n+r_n}$ into r_n equal parts;

(3_n) the points $P_{n,kq_n+r_n}, \cdots, P_{n,(k+1)q_n}$ are the vertices of a broken line whose segments are diagonals of the r_n^2 equal squares into which square $P_{n,kq_n+r_n} P_{n,(k+1)q_n}$ $(= M_{n-1,k+1} P_{n-1,k+1})$ may be divided and which is such that

$$(P_{n,kq_n+r_n}, \cdots, P_{n,(k+1)q_n})-(M_{n-1,k+1}, P_{n-1,k+1})$$
$$= \frac{2p_n(p_n+1)}{3(2p_n+1)} \cdot \text{sq. } M_{n-1,k+1} P_{n-1,k+1}.$$

Let Σ_n denote the area of the squares whose diagonals are the segments of $(P_{n0} \cdots P_{nm_n})$. Then

$$\Sigma_n = \frac{m_n}{(4p_1+2)^2 \cdots (4p_n+2)^2} = \frac{(p_1+1)\cdots(p_n+1)}{(4p_1+2)\cdots(4p_n+2)},$$

or

$$\Sigma_n = \frac{1}{4^n}\left(1+\frac{1}{2p_1+1}\right)\cdots\left(1+\frac{1}{2p_n+1}\right).$$

Hence

$$\left(\frac{1}{4}\right)^n < \Sigma_n < \left(\frac{1}{2}\right)^n.$$

We have

$$(P_{n0}, \cdots, P_{nm_n})-(P_{n-1,0}, \cdots, P_{n-1,m_{n-1}})$$
$$= \sum_{k=0}^{m_{n-1}-1} \{(P_{n,kq_n}, \cdots, P_{n,(k+1)q_n})-(P_{n-1,k} P_{n-1,k+1})\},$$

so that

$$(P_{n0} \cdots P_{nm_n})-(P_{n-1,0} \cdots P_{n-1,m_{n-1}}) = \frac{p_n(p_n+1)}{6(2p_n+1)} \Sigma_{n-1}.$$

Let l_n be the length of $P_{n0} \cdots P_{nm_n}$. Then

$$l_n = \frac{\sqrt{2}\, m_n}{(4p_1+2)\cdots(4p_n+2)} = \sqrt{2}\,(p_1+1)\cdots(p_n+1).$$

The sequence $\{p_n\}$ has so far been arbitrary. Now take p_n to be 1 plus the greatest integer in $12(4^{n-1}/n)$. Then

$$\frac{4^{n-1}}{n} \leq \frac{p_n(p_n+1)}{6(2p_n+1)} \leq \frac{1}{8}+\frac{4^{n-1}}{n}.$$

Hence

$$(P_{n0} \cdots P_{nm_n})-(P_{n-1,0} \cdots P_{n-1,m_{n-1}}) \geq \frac{4^{n-1}}{n}\cdot\left(\frac{1}{4}\right)^{n-1} = \frac{1}{n},$$

and

$$(P_{n0}\cdots P_{nm_n})-(P_{n-1,0}\cdots P_{n-1,m_{n-1}})$$
$$<\left(\frac{4^{n-1}}{n}+\frac{1}{8}\right)\left(\frac{1}{4}\right)^{n-1}\left(1+\frac{1}{2p_1+1}\right)\cdots\left(1+\frac{1}{2p_{n-1}+1}\right)$$
$$<\left(\frac{1}{n}+\frac{1}{2}\cdot\frac{1}{4^n}\right)\left(1+\frac{1}{2p_1+1}\right)\cdots\left(1+\frac{1}{2p_{n-1}+1}\right).$$

The infinite product

$$\left(1+\frac{1}{2p_1+1}\right)\left(1+\frac{1}{2p_2+1}\right)\cdots$$

converges, since the series

$$\frac{1}{2p_1+1}+\frac{1}{2p_2+1}+\cdots$$

converges, being dominated by the convergent series

$$\frac{1}{24}\left(1+\frac{2}{4}+\frac{3}{4^2}+\frac{4}{4^3}+\cdots\right).$$

Hence

$$\underset{n\to\infty}{\mathrm{L}}\{(P_{n0}\cdots P_{nm_n})-(P_{n-1,0}\cdots P_{n-1,m_{n-1}})\} = 0.$$

We note further that for every n and p,

$$(2)\qquad (P_{n+p,0}\cdots P_{n+p,m_{n+p}})-(P_{n0}\cdots P_{nm_n})\geqq\frac{1}{n+1}+\cdots+\frac{1}{n+p}.$$

Now let us write the parametric equations for the broken line $(P_{n0}\cdots P_{nm_n})$ thus:

$$x=\varphi_n(t),\qquad y=\psi_n(t)\qquad\qquad (0\leqq t\leqq 1)$$

where the parameter t is proportional to the length of arc along the broken line.

The functions $\varphi_n(t)$, $\psi_n(t)$ will converge uniformly to two continuous functions $\varphi(t)$, $\psi(t)$ respectively; it is the Stieltjes integral $\int_0^1\psi(t)\,d\varphi(t)$ of these functions which we wish to study.

Let π_n be that partition obtained by dividing (0 1) into m_n equal parts, so that $\mathrm{N}\pi_n = 1/m^n$. Then

$$\int_0^1\psi_{\pi_n}\,d\varphi_{\pi_n}=\int_0^1\psi_n\,d\varphi_n=(P_{n0}\cdots P_{nm_n}).$$

Hence by (2) $\underset{n}{\mathrm{L}}\int_0^1 \psi_{\pi_n}\, d\varphi_{\pi_n}$ does not exist and $\int_0^1 \psi\, d\varphi$ does not exist in either of the four senses.

We now investigate the conditions $O_F(\varphi\,\psi)$, $O_N(\varphi\,\psi)$. To this end, let $n(I)$ be defined by the inequality

$$\mathrm{N}\,\pi_{n(I)} \leqq I < \mathrm{N}\,\pi_{n(I)-1}.$$

We have

$$(O_I\,\psi)\,|\varphi(I)| \leqq \frac{1}{2} I_*^2\, l_{n(I)}^2,$$

where I_* is the sum of all the cells of $\pi_{n(I)}$ which have inner points in common with I. For the points $(\varphi(t), \psi(t))$ for t in I lie entirely in certain squares of the system $\Sigma_{n(I)}$. The left hand side of the inequality will have its greatest value when these squares are diagonally collinear, in which case they all lie in a square whose diagonal is $I_*\, l_{n(I)}$.

Now $I_* \leqq 3\,I$ always, hence

$$(O_I\,\psi)\,|\varphi(I)| \leqq \frac{9}{2} I^2\, l_{n(I)}^2.$$

Moreover

$$l_n^2 = 2\,\Sigma_n\, m_n = \frac{2\,\Sigma_n}{N\pi_n},$$

in view of which

$$(O_I\,\psi)\,|\varphi(I)| \leqq 9\,I^2 \frac{\Sigma_n}{N\pi_n},$$

so that

$$(O_I\,\psi)\,|\varphi(I)| \leqq 9\,[2\,p_{n(I)}+1]\,I\,\Sigma_{n(I)} \qquad (I \leqq [2\,p_{n(I)}+1]\,\mathrm{N}\,\pi_{n(I)}).$$

Now consider the set of all I for which $n(I) = n_0$. Since $I < \mathrm{N}\,\pi_{n_0-1}$, the points $(\varphi(t), \psi(t))$ for t in I lie entirely in two squares of Σ_{n_0-1} and hence $(O_I\,\psi)\,|\varphi(I)|$ is less than the area of four squares of Σ_{n_0-1}, that is, than $(4/(m_{n_0-1}))\,\Sigma_{n_0-1}$. By property (2_n) of the polygon $(P_{n_0} \cdots P_{nm_n})$, there is an $I_0 = (2\,p_{n_0}+1)\,\mathrm{N}\,\pi_{n_0}$ such that

$$(O_{I_0}\,\psi)\,|\varphi(I_0)| = \frac{1}{m_{n_0-1}}\,\Sigma_{n_0-1}.$$

Hence

$$\begin{aligned}(O_I\,\psi)\,|\varphi(I)| &\leqq 4\,(O_{I_0}\,\psi)\,|\varphi(I_0)| \\ &\leqq 36\,(2\,p_{n_0}+1)\,I_0\,\Sigma_{n_0} \\ &< 36\,(2\,p_{n_0}+1)\,I\,\Sigma_{n_0} \quad \text{if } I > (2\,p_{n_0}+1)\,\mathrm{N}\,\pi_{n_0}.\end{aligned}$$

From this it follows that for all I,

$$
\begin{aligned}
(O_I\,\psi)\,|\varphi(I)| &\leqq 36\,(2\,p_{n(I)}+1)\,I\,\Sigma_{n(I)} \\
&= \frac{216\,(p_{n(I)+1}+1)}{p_{n(I)+1}\,(p_{n(I)+1}+1)}\,I\left[\int_0^1 \psi_{n(I)+1}\,d\,\varphi_{n(I)+1}-\int_0^1 \psi_{n(I)}\,d\,\varphi_{n(I)}\right] \\
&\leqq I\,M\left[\int_0^1 \psi_{n(I)+1}\,d\,\varphi_{n(I)+1}-\int_0^1 \psi_{n(I)}\,d\,\varphi_{n(I)}\right],
\end{aligned}
$$

where M is the least upper bound of

$$\frac{216\,(2\,p_n+1)}{p_n\,(p_n+1)},$$

which is surely finite since

$$\mathop{\mathrm{L}}_{n}\frac{216\,(2\,p_n+1)}{p_n\,(p_n+1)} = 0.$$

Now denote by a_π the largest value of

$$\int_0^1 \psi_{n(\Delta\pi)+1}\,d\,\varphi_{n(\Delta\pi)+1}-\int_0^1 \psi_{n(\Delta\pi)}\,d\,\varphi_{n(\Delta\pi)}$$

for all $\Delta\pi$ of π.

Clearly

$$\mathop{\mathrm{L_F}}_{\pi} a_\pi = \mathop{\mathrm{L_N}}_{\pi} a_\pi = 0.$$

Hence since

$$\sum_{\Delta\pi}^{\pi}(O_{\Delta\pi}\,\psi)\,|\varphi(\Delta\pi)| \leqq M a_\pi,$$

we have

$$\mathrm{L_N}\,S(\Delta\,O\,\psi)\,|\Delta\varphi| = \mathrm{L_F}\,S(\Delta\,O\,\psi)\,|\Delta\varphi| = 0.$$

That is, φ, ψ satisfy the condition $A_F(\varphi\,\psi)$, $A_N(\varphi\,\psi)$, $O_F(\varphi\,\psi)$, $O_N(\varphi\,\psi)$, and since the integral fails to exist, do not satisfy the conditions $J_F(\varphi\,\psi)$, $J_N(\varphi\,\psi)$.

University of Minnesota,
Minneapolis, Minn.

THE MUTUAL INDUCTANCE OF TWO SQUARE COILS*

BY

T. H. GRONWALL

In the design of apparatus for the absolute measurement of radio field intensity, it is necessary to compute the mutual inductance of two square coils in the following position: In a vertical plane, place two squares of sides l and L, where $l < L$, so that their centers coincide and a pair of sides in each square are vertical. Rotate the square of side l through an angle α about a vertical axis passing through the midpoints of the horizontal sides, and move the square of side L through a distance h perpendicularly to the original plane. We thus arrive at the arrangement presented in top and side views in the figure (p. 519).

The mutual inductance I of two coils is given by the Neumann integral

$$I = \int\int \frac{\cos\theta}{r}\, ds\, ds',$$

where ds and ds' are the line elements of the first and second coil, respectively, θ being the angle and r the distance between ds and ds'.

In our case, the coils are made up from straight segments, and the integral may be expressed explicitly in terms of the elementary functions by means of formulas due to G. A. Campbell.† While these formulas are very interesting from a theoretical point of view, they do not work well in the present problem. In fact, L and l being given, the design of the apparatus requires the numerical values of I for several values of h and a large number of values of α, and the formulas referred to would consequently have to be evaluated separately and independently for each combination of values of h and α. Even with the aid of the graphical method proposed by Campbell, this numerical work becomes rather formidable.

It is the purpose of the present paper to give a series expansion which solves the problem of numerical computation in what appears to be the simplest manner possible under the circumstances. The number of terms written out in the formula is sufficient to give I/L with an error less than two units in the third decimal place under the condition $l < 0.35\,L$ occurring in practice (for $l = 0.35\,L$ and $h = \alpha = 0$, this error is less than 0.15 %).

* Presented to the Society, April 28, 1923. This investigation was undertaken at the request of the American Telephone and Telegraph Company, and is published with their permission.

† G. A. Campbell, *Mutual inductances of circuits composed of straight wires*, Physical Review, new ser., vol. 5, pp. 452–458; June, 1915.

The mutual inductance I of two square coils in the position shown in the figure is given by

$$\frac{I}{L} = \sum_{n=1}^{\infty} \lambda^{2n} \sum_{m=1}^{n} f_{mn}(\varkappa) \cos(2m-1)\alpha,$$

where

$$\varkappa = \frac{2h}{L}, \qquad \lambda = \frac{l}{L},$$

the series being convergent for $2\lambda^2 < 1 + \varkappa^2$. *Retaining only terms up to* $n = 3$ *inclusive, we obtain the approximate formula*

$$\begin{aligned}\frac{I}{L} = \quad & [f_{11}(\varkappa)\lambda^2 + f_{12}(\varkappa)\lambda^4 + f_{13}(\varkappa)\lambda^6]\cos\alpha \\ & + [f_{22}(\varkappa)\lambda^4 + f_{23}(\varkappa)\lambda^6]\cos 3\alpha \\ & + [f_{33}(\varkappa)\lambda^6]\cos 5\alpha\end{aligned}$$

with an error less than 0.002 *for* $\lambda < 0.35$. *Writing*

$$a = \frac{1}{1+\varkappa^2}, \qquad b = \frac{1}{\sqrt{2+\varkappa^2}},$$

we have the following values of the coefficients:

$$f_{11}(\varkappa) = 16ab,$$

$$f_{12}(\varkappa) = 3a^2\left[(-4+5a)\left(b - \frac{2}{3}b^3 + \frac{1}{5}b^5\right) + \frac{1}{3}b^3 - \frac{1}{5}b^5\right],$$

$$f_{22}(\varkappa) = \frac{5}{3}f_{12}(\varkappa),$$

$$\begin{aligned}f_{13}(\varkappa) = 7a^3\Bigg[&\left(\frac{3}{4} - \frac{49}{12}a + \frac{21}{8}a^2\right)\left(b - \frac{4}{3}b^3 + \frac{6}{5}b^5 - \frac{4}{7}b^7 + \frac{1}{9}b^9\right) \\ &+ \left(\frac{13}{4} + \frac{77}{12}a\right)\left(\frac{1}{3}b^3 - \frac{3}{5}b^5 + \frac{3}{7}b^7 - \frac{1}{9}b^9\right) \\ &- \frac{43}{8}\left(\frac{1}{5}b^5 - \frac{2}{7}b^7 + \frac{1}{9}b^9\right)\Bigg],\end{aligned}$$

$$\begin{aligned}f_{23}(\varkappa) = 7a^3\Bigg[&\left(\frac{31}{8} - \frac{245}{24}a + \frac{93}{16}a^2\right)\left(b - \frac{4}{3}b^3 + \frac{6}{5}b^5 - \frac{4}{7}b^7 + \frac{1}{9}b^9\right) \\ &+ \left(-\frac{127}{8} + \frac{565}{24}a\right)\left(\frac{1}{3}b^3 - \frac{3}{5}b^5 + \frac{3}{7}b^7 - \frac{1}{9}b^9\right) \\ &- \frac{19}{16}\left(\frac{1}{5}b^5 - \frac{2}{7}b^7 + \frac{1}{9}b^9\right)\Bigg],\end{aligned}$$

$$f_{33}(\varkappa) = 63a^3\left[\left(\frac{3}{8} - \frac{11}{8}a + \frac{17}{16}a^2\right)\left(b - \frac{4}{3}b^3 + \frac{6}{5}b^5 - \frac{4}{7}b^7 + \frac{1}{9}b^9\right)\right.$$
$$+\left(-\frac{3}{8} + \frac{3}{8}a\right)\left(\frac{1}{3}b^3 - \frac{3}{5}b^5 + \frac{3}{7}b^7 - \frac{1}{9}b^9\right)$$
$$\left.+\frac{1}{16}\left(\frac{1}{5}b^5 - \frac{2}{7}b^7 + \frac{1}{9}b^9\right)\right].$$

PROOF

1. **The Neumann integral.** Consider first the vertical segments; letting $\overline{AC}$ be the horizontal distance between the two segments 2 and 6, it is seen from the top view in the figure that

$$\overline{AC}^2 = (\tfrac{1}{2}L - \tfrac{1}{2}l\cos\alpha)^2 + (h - \tfrac{1}{2}l\sin\alpha)^2.$$

Measure x' and y' along segments 2 and 6, respectively, from their midpoints and positive in the direction of the arrow; then the distance between the two corresponding points is given by

$$r_{26}^2 = (\tfrac{1}{2}L - \tfrac{1}{2}l\cos\alpha)^2 + (h - \tfrac{1}{2}l\sin\alpha)^2 + (x' - y')^2.$$

Similarly

$$r_{28}^2 = (\tfrac{1}{2}L + \tfrac{1}{2}l\cos\alpha)^2 + (h + \tfrac{1}{2}l\sin\alpha)^2 + (x' + y')^2,$$
$$r_{46}^2 = (\tfrac{1}{2}L + \tfrac{1}{2}l\cos\alpha)^2 + (h - \tfrac{1}{2}l\sin\alpha)^2 + (x' + y')^2,$$
$$r_{48}^2 = (\tfrac{1}{2}L - \tfrac{1}{2}l\cos\alpha)^2 + (h + \tfrac{1}{2}l\sin\alpha)^2 + (x' - y')^2.$$

Since $\theta = 0$ for the segments 2,6 and 4,8, but $\theta = \pi$ for the segments 2,8 and 4,6, while $\theta = \pi/2$ for a vertical and a horizontal segment, it follows that the contribution of the vertical segments to the Neumann integral is

$$I' = \int_{-\frac{1}{2}L}^{\frac{1}{2}L} dx' \int_{-\frac{1}{2}l}^{\frac{1}{2}l} dy' \left(\frac{1}{r_{26}} - \frac{1}{r_{28}} + \frac{1}{r_{48}} - \frac{1}{r_{46}}\right).$$

Introducing the notations

(1) $$\varkappa = 2h/L, \qquad \lambda = l/L,$$

and substituting $x' = Lx/2$, $y' = ly/2 = L\lambda y/2$ in the integral, we obtain

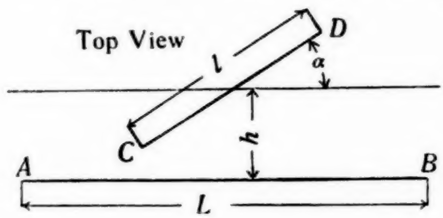
(2) $$I' = \tfrac{1}{2}L[F(\alpha, \varkappa, \lambda) + F(\alpha, -\varkappa, \lambda) + F(\alpha, \varkappa, -\lambda) + F(\alpha, -\varkappa, -\lambda)],$$

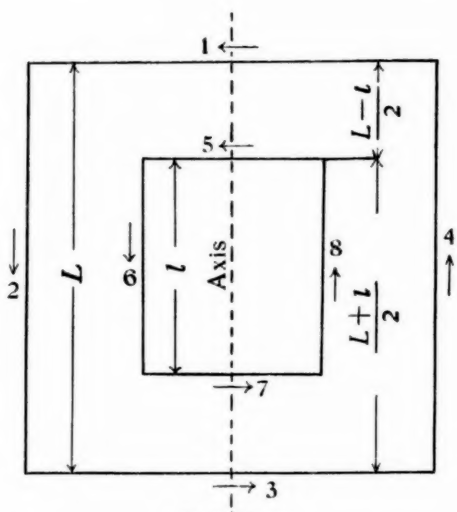
where

(3) $$F(\alpha, \varkappa, \lambda) = \int_{-1}^{1}\int_{-1}^{1} \frac{\lambda}{V}\, dx\, dy$$

and

(4) $$V^2 = (1-\lambda\cos\alpha)^2 + (\varkappa - \lambda\sin\alpha)^2 + (x - \lambda y)^2.$$





Turning our attention to the horizontal segments, let us measure x' and y' along segments 1 and 5, respectively, from their midpoints and positive in the direction of the arrow; it is then seen from the figure that the distance r_{15} between the two corresponding points is given by

$$r_{15}^2 = (\tfrac{1}{2}L - \tfrac{1}{2}l)^2 + (x' - y'\cos\alpha)^2 + (h - y'\sin\alpha)^2,$$

and similarly

$$r_{17}^2 = (\tfrac{1}{2}L + \tfrac{1}{2}l)^2 + (x' + y'\cos\alpha)^2 + (h + y'\sin\alpha)^2,$$

$$r_{35}^2 = r_{17}^2, \qquad r_{37}^2 = r_{15}^2.$$

Since $\theta=\alpha$ for the segments 1,5 and 3,7, but $\theta=\pi+\alpha$ for the segments 1,7 and 3,5, it follows that the contribution of the horizontal segments to the Neumann integral is

$$I''=\int_{-\frac{1}{2}L}^{\frac{1}{2}L}dx'\int_{-\frac{1}{2}l}^{\frac{1}{2}l}dy'\left(\frac{2\cos\alpha}{r_{15}}-\frac{2\cos\alpha}{r_{17}}\right).$$

By the same substitution as before, we obtain

$$I''=L[G(\alpha,\varkappa,\lambda)+G(\alpha,\varkappa,-\lambda)],$$

where

$$G(\alpha,\varkappa,\lambda)=\int_{-1}^{1}\int_{-1}^{1}\frac{\lambda\cos\alpha}{H}\,dx\,dy, \tag{5}$$

and

$$H^2=(1-\lambda)^2+(x-\lambda y\cos\alpha)^2+(\varkappa-\lambda y\sin\alpha)^2. \tag{6}$$

We have $G(\alpha,-\varkappa,\lambda)=G(\alpha,\varkappa,\lambda)$, as is seen from (5) and (6) upon changing x, y and $\varkappa$ into $-x$, $-y$ and $-\varkappa$ respectively. Thus the formula for I'' may be written in a form exactly similar to (2):

$$I''=\tfrac{1}{2}L[G(\alpha,\varkappa,\lambda)+G(\alpha,-\varkappa,\lambda)+G(\alpha,\varkappa,-\lambda)+G(\alpha,-\varkappa,-\lambda)]. \tag{7}$$

2. **Expansion in powers of λ.** From (4) and (6), we have

$$V^2=1+\varkappa^2+x^2-2(\cos\alpha+\varkappa\sin\alpha+xy)\lambda+(1+y^2)\lambda^2,$$
$$H^2=1+\varkappa^2+x^2-2(1+xy\cos\alpha+\varkappa y\sin\alpha)\lambda+(1+y^2)\lambda^2,$$

and consequently

$$\begin{aligned}\frac{1}{V}&=(1+\varkappa^2+x^2)^{-1/2}(1-2\xi z+z^2)^{-1/2},\\ \frac{1}{H}&=(1+\varkappa^2+x^2)^{-1/2}(1-2\eta z+z^2)^{-1/2},\end{aligned} \tag{8}$$

where

$$\begin{aligned}\xi&=\frac{\cos\alpha+\varkappa\sin\alpha+xy}{(1+\varkappa^2+x^2)^{1/2}(1+y^2)^{1/2}},\\ \eta&=\frac{1+xy\cos\alpha+\varkappa y\sin\alpha}{(1+\varkappa^2+x^2)^{1/2}(1+y^2)^{1/2}},\\ z&=\lambda\left(\frac{1+y^2}{1+\varkappa^2+x^2}\right)^{1/2}.\end{aligned} \tag{9}$$

We now note the relations

$$(1+\varkappa^2+x^2)(1+y^2)-(\cos\alpha+\varkappa\sin\alpha+xy)^2$$
$$=(\sin\alpha-\varkappa\cos\alpha)^2+(x\cos\alpha-y)^2+(x\sin\alpha-\varkappa y)^2\geqq 0,$$
$$(1+\varkappa^2+x^2)(1+y^2)-(1+xy\cos\alpha+\varkappa y\sin\alpha)^2$$
$$=(xy\sin\alpha-\varkappa y\cos\alpha)^2+(\varkappa-y\sin\alpha)^2+(x-y\cos\alpha)^2\geqq 0,$$

from which it follows that $\xi^2\leqq 1$, $\eta^2\leqq 1$, or

$$\text{(10)}\qquad -1\leqq\xi\leqq 1,\qquad -1\leqq\eta\leqq 1.$$

From the last of (9), it is seen that, for $-1\leqq x\leqq 1$, $-1\leqq y\leqq 1$, and for

$$\text{(11)}\qquad 2\lambda^2<1+\varkappa^2$$

we have

$$\text{(12)}\qquad |z|<|\lambda|\left(\frac{2}{1+\varkappa^2}\right)^{1/2}<1,$$

and in consequence of (10), (11) and (12), the expansions

$$(1-2\xi z+z^2)^{-1/2}=\sum_{n=0}^{\infty}P_n(\xi)z^n,$$
$$(1-2\eta z+z^2)^{-1/2}=\sum_{n=0}^{\infty}P_n(\eta)z^n,$$

where P_n are the Legendre polynomials, converge uniformly for $-1\leqq x\leqq 1$, $-1\leqq y\leqq 1$. Consequently, we may introduce these expansions in the formulas for $F(\alpha,\varkappa,\lambda)$ and $G(\alpha,\varkappa,\lambda)$ and integrate term by term, thus obtaining

$$F(\alpha,\varkappa,\lambda)=\sum_{n=0}^{\infty}\int_{-1}^{1}\int_{-1}^{1}\frac{\lambda P_n(\xi)z^n}{(1+\varkappa^2+x^2)^{1/2}}\,dx\,dy,$$
$$G(\alpha,\varkappa,\lambda)=\sum_{n=0}^{\infty}\int_{-1}^{1}\int_{-1}^{1}\frac{\lambda\cos\alpha\, P_n(\eta)z^n}{(1+\varkappa^2+x^2)^{1/2}}\,dx\,dy.$$

Changing λ into $-\lambda$, it is seen from (9) that z changes into $-z$, while ξ and η remain the same, being independent of λ. Consequently

$$F(\alpha,\varkappa,\lambda)+F(\alpha,\varkappa,-\lambda)=\sum_{n=0}^{\infty}\int_{-1}^{1}\int_{-1}^{1}\frac{[1+(-1)^n]\lambda P_n(\xi)z^n}{(1+\varkappa^2+x^2)^{1/2}}\,dx\,dy$$

with a similar formula for G, or

$$F(\alpha, \varkappa, \lambda) + F(\alpha, \varkappa, -\lambda) = 2\sum_{n=0}^{\infty}\int_{-1}^{1}\int_{-1}^{1}\frac{\lambda\, P_{2n+1}(\xi)\, z^{2n+1}}{(1+\varkappa^2+x^2)^{1/2}}\,dx\,dy,$$

$$G(\alpha, \varkappa, \lambda) + G(\alpha, \varkappa, -\lambda) = 2\sum_{n=0}^{\infty}\int_{-1}^{1}\int_{-1}^{1}\frac{\lambda\cos\alpha\, P_{2n+1}(\eta)\, z^{2n+1}}{(1+\varkappa^2+x^2)^{1/2}}\,dx\,dy.$$

Consequently, by (2) and (5),

$$\frac{I}{L} = \frac{I'}{L} + \frac{I''}{L} = \sum_{n=1}^{\infty}\varphi_n(\alpha, \varkappa)\lambda^{2n}, \tag{13}$$

where

(14) $\varphi_n(\alpha, \varkappa)$

$$= \int_{-1}^{1}\int_{-1}^{1}\frac{[P_{2n-1}(\xi)+P_{2n-1}(\xi_1)+\cos\alpha P_{2n-1}(\eta)+\cos\alpha P_{2n-1}(\eta_1)](1+y^2)^{n-1/2}}{(1+\varkappa^2+x^2)^n}\,dx\,dy$$

and ξ_1 and η_1 are obtained from ξ and η in (9) by changing $\varkappa$ into $-\varkappa$, the series (13) being convergent under the condition (11).

3. **Upper bound for the absolute value of the remainder in the series.** Before evaluating the expression (14), it is appropriate to form an estimate of the error R_p committed by retaining in (13) only the first p terms, that is, only terms containing powers of λ up to the $2p$th inclusive. This error is evidently the series remainder

$$R_p = \sum_{n=p}^{\infty}\varphi_{n+1}(\alpha, \varkappa)\lambda^{2n+2}, \tag{15}$$

so that

$$|R_p| \leqq \sum_{n=p}^{\infty}|\varphi_{n+1}(\alpha, \varkappa)|\,\lambda^{2n+2}. \tag{16}$$

Now ξ, ξ_1, η and η_1 all lie in the interval from -1 to $+1$ by (10), and consequently every Legendre polynomial in one of these four arguments does not exceed unity in absolute value, so that (14) gives

$$|\varphi_{n+1}(\alpha, \varkappa)| \leqq 2(1+|\cos\alpha|)\int_{-1}^{1}\int_{-1}^{1}\frac{(1+y^2)^{n+1/2}}{(1+\varkappa^2+x^2)^{n+1}}\,dx\,dy. \tag{17}$$

To get an upper bound for the expression to the right in (17), we observe that

$$\int_{-1}^{1}\frac{dx}{(1+\varkappa^2+x^2)^{n+1}} < \int_{-\infty}^{\infty}\frac{dx}{(1+\varkappa^2+x^2)^{n+1}}$$
$$= \frac{1\cdot 3\cdot 5\cdots(2n-1)}{2\cdot 4\cdot 6\cdots 2n}\cdot\frac{\pi}{(1+\varkappa^2)^{n+1/2}},$$

the last integral being obtained by differentiating the integral

$$\int_{-\infty}^{\infty}\frac{dx}{1+\varkappa^2+x^2}=\left[\frac{1}{(1+\varkappa^2)^{1/2}}\operatorname{arc\,tan}\frac{x}{(1+\varkappa^2)^{1/2}}\right]_{-\infty}^{\infty}=\frac{\pi}{(1+\varkappa^2)^{1/2}}$$

n times in respect to $\varkappa^2$. Consequently we have

$$(18)\quad \int_{-1}^{1}\frac{dx}{(1+\varkappa^2+x^2)^{n+1}}<\frac{1\cdot 3\cdot 5\cdots(2p-1)}{2\cdot 4\cdot 6\cdots 2p}\cdot\frac{\pi}{(1+\varkappa^2)^{n+1/2}}\text{ for } n\geqq p.$$

We also need a sufficiently close upper bound for $\int_{-1}^{1}(1+y^2)^{n+1/2}dy$; the identity

$$\frac{d[y(1+y^2)^{n+1/2}]}{dy}=(2n+2)(1+y^2)^{n+1/2}-(2n+1)(1+y^2)^{n-1/2}$$

yields the reduction formula

$$(19)\quad \int_{-1}^{1}(1+y^2)^{n+1/2}dy=\frac{2^n\sqrt{2}}{n+1}+\frac{2n+1}{2n+2}\int_{-1}^{1}(1+y^2)^{n-1/2}dy,$$

from which we readily obtain

$$\int_{-1}^{1}(1+y^2)^{4+1/2}dy=\frac{4607\sqrt{2}}{640}+\frac{63}{256}\log(1+\sqrt{2})=10.397.$$

It will now be shown that

$$(20)\qquad \int_{-1}^{1}(1+y^2)^{n+1/2}dy<\frac{2^{n+1}\cdot 3\sqrt{2}}{3n+1}\text{ for } n\geqq 4.$$

In fact, for $n=4$ the expression to the right equals 10.443, so that (20) holds in this case, and further numerical computation by means of (19) shows that the inequality is verified also for $n=5$, 6 and 7. Now assume (20) to be true for the exponent $n-1$; then (19) gives

$$\int_{-1}^{1}(1+y^2)^{n+1/2}dy<\frac{2^n\sqrt{2}}{n+1}+\frac{2n+1}{2n+2}\cdot\frac{2^n\cdot 3\sqrt{2}}{3n-2},$$

and consequently

$$\begin{aligned}\int_{-1}^{1}(1+y^2)^{n+1/2}dy&-\frac{2^{n+1}\cdot 3\sqrt{2}}{3n+1}\\&<2^{n-1}\sqrt{2}\left[\frac{2}{n+1}+\frac{3(2n+1)}{(n+1)(3n-2)}-\frac{12}{3n+1}\right]\\&=\frac{2^{n-1}\sqrt{2}\,(23-3n)}{(n+1)(3n-2)(3n+1)}<0\text{ for } n\geqq 8,\end{aligned}$$

so that (20) is true for all $n\geqq 8$, which completes the proof.

Introducing (18) and (20) in (17), it follows from (16) that

$$|R_p| < 2(1+|\cos\alpha|)\frac{1\cdot 3\cdot 5\cdots(2p-1)\cdot 3\pi\sqrt{2}}{2\cdot 4\cdot 6\cdots 2p}\sum_{n=p}^{\infty}\frac{1}{3n+1}\,\frac{2^{n+1}\lambda^{2n+2}}{(1+\varkappa^2)^{n+1/2}},$$

or replacing $3n+1$ by $3p+1$, which increases each term in the series except the first, and summing the resulting geometric series,

$$(21)\quad |R_p| < (1+|\cos\alpha|)\frac{1\cdot 3\cdot 5\cdots(2p-1)}{2\cdot 4\cdot 6\cdots 2p}\cdot\frac{6\pi\sqrt{2(1+\varkappa^2)}}{3p+1}\cdot\frac{\left(\dfrac{2\lambda^2}{1+\varkappa^2}\right)^{p+1}}{1-\dfrac{2\lambda^2}{1+\varkappa^2}},$$

which is valid for $p \geqq 4$ and constitutes the desired upper bound for the absolute value of the error committed in neglecting all powers of λ higher than the $2p$th.

It is evident from (21) that the right hand member increases with $|\cos\alpha|$, but decreases as $\varkappa$ increases. Making $\alpha = 0$, $\varkappa = 0$ and $p = 4$, we therefore have

$$(22)\qquad |R_4| < 2\cdot\frac{1\cdot 3\cdot 5\cdot 7}{2\cdot 4\cdot 6\cdot 8}\cdot\frac{6\pi\sqrt{2}}{13}\cdot\frac{(2\lambda^2)^5}{1-2\lambda^2} = 1.122\cdot\frac{(2\lambda^2)^5}{1-2\lambda^2}$$

for all values of α and $\varkappa$.

For purposes of comparison, the exact values of I/L and R_4 were computed from (65) for $\alpha = 0$, $\varkappa = 0$ and $\lambda = 0.30$, 0.35 and 0.40, as well as the upper bound of $|R_4|$ from (22), with the following results:

λ	I/L	R_4 (exact)	$\lvert R_4\rvert$ by (22)
0.30	1.05870	$+0.00001$	<0.00026
0.35	1.46258	$+0.00005$	<0.00131
0.40	1.94440	$+0.00021$	<0.00553

4. **Expansion in cosines of the odd multiples of α.** We shall now expand each $\varphi_n(\alpha, \varkappa)$ as given by (14) in a finite Fourier series of the form

$$(23)\qquad \varphi_n(\alpha, \varkappa) = \sum_{m=1}^{n} f_{mn}(\varkappa)\cos(2m-1)\alpha.$$

To this purpose, we introduce the associated Legendre polynomials

$$(24)\qquad P_n^m(z) = (1-z^2)^{m/2}\cdot\frac{d^m P_n(z)}{dz^m},$$

with their representation by an integral of the Laplace type

$$(25)\qquad P_n^m(z) = \frac{(n+m)!}{n!\,2\pi i^m}\int_{-\pi}^{\pi}(z+i\sqrt{1-z^2})^n\cos m\varphi\,d\varphi,$$

and finally the addition theorem for the Legendre polynomials

$$(26)\qquad \begin{aligned} P_n(\cos\theta\cos\theta'+\sin\theta\sin\theta'\cos\omega) &= P_n(\cos\theta)\,P_n(\cos\theta') \\ &+2\sum_{m=1}^{n}\frac{(n-m)!}{(n+m)!}P_n^m(\cos\theta)\,P_n^m(\cos\theta')\cos m\omega, \end{aligned}$$

where $0\leq\theta\leq\pi$ and $0\leq\theta'\leq\pi$.*

Defining β by the equations

$$(27)\qquad \cos\beta = \frac{1}{(1+\varkappa^2)^{1/2}},\qquad \sin\beta = \frac{\varkappa}{(1+\varkappa^2)^{1/2}},$$

and making

$$(28)\qquad \begin{aligned} \cos\theta &= \frac{x}{(1+\varkappa^2+x^2)^{1/2}}, & \sin\theta &= \frac{(1+\varkappa^2)^{1/2}}{(1+\varkappa^2+x^2)^{1/2}}, \\ \cos\theta' &= \frac{y}{(1+y^2)^{1/2}}, & \sin\theta' &= \frac{1}{(1+y^2)^{1/2}}, \end{aligned}$$

we find from (9) that

$$\begin{aligned} \xi &= \cos\theta\cos\theta'+\sin\theta\sin\theta'\cos(\alpha-\beta), \\ \xi_1 &= \cos\theta\cos\theta'+\sin\theta\sin\theta'\cos(\alpha+\beta); \end{aligned}$$

using also the formula $\cos(\alpha-\beta)+\cos(\alpha+\beta) = 2\cos\alpha\cos\beta$, we obtain from (26)

$$\begin{aligned} P_{2n-1}(\xi)+P_{2n-1}(\xi_1) &= 2P_{2n-1}(\cos\theta)\,P_{2n-1}(\cos\theta') \\ &+4\sum_{m=1}^{2n-1}\left[\frac{(2n-1-m)!}{(2n-1+m)!}P_{2n-1}^m(\cos\theta)\,P_{2n-1}^m(\cos\theta')\cos m\beta\cos m\alpha\right]. \end{aligned}$$

Changing y into $-y$, it is seen that

$$\int_{-1}^{1}P_{2n-1}^m(\cos\theta')\,(1+y^2)^{n-1/2}\,dy = \int_{-1}^{1}P_{2n-1}^m(-\cos\theta')\,(1+y^2)^{n-1/2}\,dy,$$

* For these formulas, see for instance Whittaker and Watson, *Modern Analysis*, 2d edition, pp. 317–322. It should be noted that in establishing (25) and (26), these authors use a function P_n^m which is the one in (24) multiplied by i^m.

and by (24), $P_{2n-1}^{m}(-z) = (-1)^{2n-1-m} P_{2n-1}^{m}(z)$, so that the integral vanishes when m is even; consequently

$$
\begin{aligned}
(29)\quad \int_{-1}^{1}\int_{-1}^{1} \frac{[P_{2n-1}(\xi) + P_{2n-1}(\xi_1)]\,(1+y^2)^{n-1/2}}{(1+\varkappa^2+x^2)^n}\,dx\,dy \\
= \sum_{m=1}^{n} g_{mn}(\varkappa)\cos(2m-1)\alpha,
\end{aligned}
$$

where

$$
\begin{aligned}
g_{mn}(\varkappa) = \frac{4\cdot(2n-2m)!}{(2n+2m-2)!}\cos(2m-1)\beta \\
(30)\quad \times \int_{-1}^{1} P_{2n-1}^{2m-1}\left(\frac{x}{(1+\varkappa^2+x^2)^{1/2}}\right)\frac{dx}{(1+\varkappa^2+x^2)^n} \\
\times \int_{-1}^{1} P_{2n-1}^{2m-1}\left(\frac{y}{(1+y^2)^{1/2}}\right)(1+y^2)^{n-1/2}\,dy.
\end{aligned}
$$

Before proceeding further, we shall evaluate the integral

$$
\int_{-\pi}^{\pi}(\cos\varphi)^a\cos b\varphi\,d\varphi,
$$

where a and b are positive integers or zero. Since the corresponding integral with $\sin b\varphi$ instead of $\cos b\varphi$ vanishes, the integrand being odd, our integral is equal to

$$
\begin{aligned}
\int_{-\pi}^{\pi}(\cos\varphi)^a e^{b\varphi i}\,d\varphi &= \int_{-\pi}^{\pi}\left(\frac{e^{\varphi i}+e^{-\varphi i}}{2}\right)^a e^{b\varphi i}\,d\varphi \\
&= \sum_{p=0}^{a}\left[\frac{a!}{2^a p!\,(a-p)!}\int_{-\pi}^{\pi} e^{(2p-a+b)\varphi i}\,d\varphi\right],
\end{aligned}
$$

and in the last expression, every integral vanishes for which $2p-a+b \neq 0$. Consequently

$$
\begin{aligned}
(31)\quad \int_{-\pi}^{\pi}(\cos\varphi)^a\cos b\varphi\,d\varphi &= \frac{\pi}{2^{a-1}}\cdot\frac{a!}{((a-b)/2)!\,((a+b)/2)!} \\
&\qquad \text{for } b \leq a \text{ and } a-b \text{ even}, \\
&= 0 \text{ in all other cases.}
\end{aligned}
$$

The last factor in (30) is now readily evaluated by means of (25), which gives

$$
\begin{aligned}
A_{mn} &= \int_{-1}^{1} P_{2n-1}^{2m-1}\left(\frac{y}{(1+y^2)^{1/2}}\right)(1+y^2)^{n-1/2}\,dy \\
&= \frac{(2n+2m-2)!}{(2n-1)!\,2\pi\, i^{2m-1}}\int_{-1}^{1}dy\int_{-\pi}^{\pi}(y+i\cos\varphi)^{2n-1}\cos(2m-1)\varphi\, d\varphi.
\end{aligned} \tag{32}
$$

Expanding $(y+i\cos\varphi)^{2n-1}$ by the binomial theorem and using (31), we readily find

$$
A_{mn} = \sum_{p=m}^{n} \frac{(-1)^{p-m}(2n+2m-2)!}{2^{2p-2}(p-m)!\,(p+m-1)!\,(2n-2p+1)!}. \tag{33}
$$

Incidentally it may be noted that, if we integrate first with respect to y in the double integral in (32), and then apply (25), we obtain

$$
A_{mn} = \frac{2^{n+1}}{2n+2m-1} P_{2n}^{2m-1}\left(\frac{1}{\sqrt{2}}\right).
$$

In the remaining integral in (30), we introduce the θ given by (28) as integration variable and obtain by (25)

$$
\begin{aligned}
&\int_{-1}^{1} P_{2n-1}^{2m-1}\left(\frac{x}{(1+\varkappa^2+x^2)^{1/2}}\right)\frac{dx}{(1+\varkappa^2+x^2)^{n}} \\
&= \int_{\gamma}^{\pi-\gamma} P_{2n-1}^{2m-1}(\cos\theta)(\cos\beta)^{2n-1}(\sin\theta)^{2n-2}\,d\theta \\
&= \frac{(2n+2m-2)!\,(\cos\beta)^{2n-1}}{(2n-1)!\,2\pi\, i^{2m-1}}\int_{\gamma}^{\pi-\gamma}(\sin\theta)^{2n-2}\,d\theta \\
&\qquad\times\int_{-\pi}^{\pi}(\cos\theta+i\sin\theta\cos\varphi)^{2n-1}\cos(2m-1)\varphi\, d\varphi
\end{aligned} \tag{34}
$$

where γ is determined by

$$
\cos\gamma = \frac{1}{(2+\varkappa^2)^{1/2}},\quad \sin\gamma = \frac{(1+\varkappa^2)^{1/2}}{(2+\varkappa^2)^{1/2}}, \tag{35}
$$

so that

$$
\tan\gamma = \sec\beta. \tag{36}
$$

Expanding by the binomial theorem under the last integration sign in (34) and using (31), it is seen that

$$
\begin{aligned}
&\int_{-1}^{1} P_{2n-1}^{2m-1}\left(\frac{x}{(1+\varkappa^2+x^2)^{1/2}}\right)\frac{dx}{(1+\varkappa^2+x^2)^{n}} \\
&= (\cos\beta)^{2n-1}\sum_{p=0}^{n-m}\left(\frac{(-1)^{n-m-p}(2n+2m-2)!}{2^{2n-2p-2}(2p)!\,(n-m-p)!\,(n+m-p-1)!}\,\psi_{np}(\gamma)\right),
\end{aligned} \tag{37}
$$

where

$$(38)\qquad \begin{aligned}\psi_{np}(\gamma) &= \frac{1}{2}\int_{\gamma}^{\pi-\gamma}(\cos\theta)^{2p}(\sin\theta)^{4n-2p-3}\,d\theta\\ &= \frac{1}{2}\int_{\gamma}^{\pi-\gamma}(\cos\theta)^{2p}(1-\cos^2\theta)^{2n-p-2}\,d\cos\theta,\end{aligned}$$

or expanding by the binomial theorem and integrating

$$(39)\qquad \psi_{np}(\gamma) = \sum_{s=0}^{2n-p-2}\left(\frac{(-1)^s(2n-p-2)!}{s!(2n-p-s-2)!}\cdot\frac{(\cos\gamma)^{2p+2s+1}}{2p+2s+1}\right).$$

From (30), (32), (37) and (38) it now follows that

$$(40)\qquad \begin{aligned}g_{mn}(\varkappa) &= 16A_{mn}\cos(2m-1)\beta(\cos\beta)^{2n-1}\\ &\quad\times\sum_{p=0}^{n-m}\left(\frac{(-1)^{n-m-p}(2n-2m)!}{2^{2n-2p}(2p)!(n-m-p)!(n+m-p-1)!}\psi_{np}(\gamma)\right).\end{aligned}$$

To deal in a similar manner with the terms in (14) which contain η and η_1, we make

$$(41)\qquad \begin{aligned}\cos\tau &= \frac{x}{(\varkappa^2+x^2)^{1/2}}, & \sin\tau &= \frac{\varkappa}{(\varkappa^2+x^2)^{1/2}},\\ \cos\theta_1 &= \frac{1}{(1+\varkappa^2+x^2)^{1/2}}, & \sin\theta_1 &= \frac{(\varkappa^2+x^2)^{1/2}}{(1+\varkappa^2+x^2)^{1/2}},\\ \cos\theta_1' &= \frac{1}{(1+y^2)^{1/2}}, & \sin\theta_1' &= \frac{|y|}{(1+y^2)^{1/2}}.\end{aligned}$$

It now follows from (9) that for $y>0$

$$\begin{aligned}\eta &= \cos\theta_1\cos\theta_1'+\sin\theta_1\sin\theta_1'\cos(\alpha-\tau),\\ \eta_1 &= \cos\theta_1\cos\theta_1'+\sin\theta_1\sin\theta_1'\cos(\alpha+\tau),\end{aligned}$$

while for $y<0$

$$\begin{aligned}\eta &= \cos\theta_1\cos\theta_1'+\sin\theta_1\sin\theta_1'\cos(\alpha-\tau+\pi),\\ \eta_1 &= \cos\theta_1\cos\theta_1'+\sin\theta_1\sin\theta_1'\cos(\alpha+\tau+\pi).\end{aligned}$$

From (26), it is now seen that

$$\int_{-1}^{1} P_{2n-1}(\eta)(1+y^2)^{n-1/2}\,dy = P_{2n-1}(\cos\theta_1)\int_{-1}^{1} P_{2n-1}(\cos\theta_1')(1+y^2)^{n-1/2}\,dy$$
$$+\,2\sum_{m=1}^{2n-1}\Bigg\{\frac{(2n-1-m)!}{(2n-1+m)!}\,P_{2n-1}^{m}(\cos\theta_1)\cos m(\alpha-\tau)$$
$$\times\Big[\int_{0}^{1} P_{2n-1}^{m}(\cos\theta_1')\,(1+y^2)^{n-1/2}\,dy$$
$$+(-1)^m\int_{-1}^{0} P_{2n-1}^{m}(\cos\theta_1')\,(1+y^2)^{n-1/2}\,dy\Big]\Bigg\},$$

and since the integrand in the last two integrals is even,

$$\int_{-1}^{1} P_{2n-1}(\eta)(1+y^2)^{n-1/2}\,dy = B_{0n}\,P_{2n-1}(\cos\theta_1)$$
$$(42)\qquad +\,2\sum_{m=1}^{n-1}\Big[\frac{(2n-2m-1)!}{(2n+2m-1)!}\,B_{mn}\,P_{2n-1}^{2m}(\cos\theta_1)\cos 2m(\alpha-\tau)\Big],$$

where

$$B_{mn} = \int_{-1}^{1} P_{2n-1}^{2m}\Big(\frac{1}{(1+y^2)^{1/2}}\Big)(1+y^2)^{n-1/2}\,dy$$
$$(43)\qquad = \frac{(2n+2m-1)!}{(2n-1)!\,2\pi i^{2m}}\int_{-1}^{1} dy\int_{-\pi}^{\pi}(1+iy\cos\varphi)^{2n-1}\cos 2m\varphi\,d\varphi,$$

or expanding by the binomial theorem and applying (31),

$$(44)\qquad B_{mn} = \sum_{p=m}^{n-1}\Big(\frac{(-1)^{p-m}(2n+2m-1)!}{(2p+1)\,2^{2p-1}(p-m)!\,(p+m)!\,(2n-2p-1)!}\Big).$$

By the aid of the formula $\cos(\alpha-\beta)+\cos(\alpha+\beta) = 2\cos\alpha\cos\beta$, (42) gives

$$\int_{-1}^{1}\int_{-1}^{1}\frac{[P_{2n-1}(\eta)+P_{2n-1}(\eta_1)]\,(1+y^2)^{n-1/2}}{(1+\varkappa^2+x^2)^n}\,dx\,dy$$
$$(45)\qquad = h_{0n}(\varkappa) + 2\sum_{m=1}^{n-1} h_{mn}(\varkappa)\cos 2m\alpha,$$

where

$$(46)\qquad h_{mn}(\varkappa) = \frac{(2n-2m-1)!}{(2n+2m-1)!}\,B_{mn}\int_{-1}^{1} 2\,P_{2n-1}^{2m}(\cos\theta_1)\,\frac{\cos 2m\tau}{(1+\varkappa^2+x^2)^n}\,dx.$$

From (14), (29) and (45) it now follows that (23) is true, the value of $f_{mn}(\varkappa)$ being

$$(47)\qquad f_{mn}(\varkappa) = g_{mn}(\varkappa) + h_{m-1,n}(\varkappa) + h_{mn}(\varkappa),$$

where, for $m = n$, we make $h_{n,n}(\varkappa) = 0$.

To evaluate (46), we use (25) and the formula $2\cos 2m\varphi\cos 2m\tau = \cos 2m(\varphi-\tau)+\cos 2m(\varphi+\tau)$, and obtain

$$
\begin{aligned}
&2P_{2n-1}^{2m}(\cos\theta_1)\cos 2m\tau \\
&= \frac{(-1)^m(2n+2m-1)!}{(2n-1)!\,2\pi}\left[\int_{-\pi}^{\pi}(\cos\theta_1+i\sin\theta_1\cos\varphi)^{2n-1}\cos 2m(\varphi+\tau)\,d\varphi\right. \\
&\qquad\left.+\int_{-\pi}^{\pi}(\cos\theta_1+i\sin\theta_1\cos\varphi)^{2n-1}\cos 2m(\varphi-\tau)\,d\varphi\right].
\end{aligned}
$$

Since the integrand has the period 2π, we may replace φ by $\varphi-\tau+\frac{1}{2}\pi$ in the first and by $\varphi+\tau+\frac{1}{2}\pi$ in the second of these integrals without changing the limits of integration, and consequently

$$
\begin{aligned}
&2P_{2n-1}^{2m}(\cos\theta_1)\cos 2m\tau \\
&\quad= \frac{(2n+2m-1)!}{(2n-1)!\,2\pi}\left[\int_{-\pi}^{\pi}(\cos\theta_1-i\sin\theta_1\sin(\varphi-\tau))^{2n-1}\cos 2m\varphi\,d\varphi\right. \\
&\qquad\left.+\int_{-\pi}^{\pi}(\cos\theta_1-i\sin\theta_1\sin(\varphi+\tau))^{2n-1}\cos 2m\varphi\,d\varphi\right].
\end{aligned}
$$

But from (41) and (28) we find

$$
\begin{aligned}
\cos\theta_1-i\sin\theta_1\sin(\varphi-\tau) &= \frac{1-ix\sin\varphi+i\varkappa\cos\varphi}{(1+\varkappa^2+x^2)^{1/2}} \\
&= (\cos\beta+i\sin\beta\cos\varphi)\sin\theta-i\sin\varphi\cos\theta, \\
\cos\theta_1-i\sin\theta_1\sin(\varphi+\tau) &= (\cos\beta-i\sin\beta\cos\varphi)\sin\theta-i\sin\varphi\cos\theta,
\end{aligned}
$$

so that, applying the binomial theorem,

$$
\begin{aligned}
&2P_{2n-1}^{2m}(\cos\theta_1)\cos 2m\tau \\
&\quad= \frac{(2n+2m-1)!}{(2n-1)!\,2\pi}\sum_{p=0}^{2n-1}\left\{\frac{(2n-1)!(-i)^p}{p!(2n-1-p)!}(\cos\theta)^p(\sin\theta)^{2n-1-p}\right. \\
&\qquad\times\int_{-\pi}^{\pi}[(\cos\beta+i\sin\beta\cos\varphi)^{2n-1-p} \\
&\qquad\left.+(\cos\beta-i\sin\beta\cos\varphi)^{2n-1-p}](\sin\varphi)^p\cos 2m\varphi\,d\varphi\right\}.
\end{aligned}
$$

The integral vanishes when p is odd, as is seen upon replacing φ by $-\varphi$, and consequently

$$2 P_{2n-1}^{2m}(\cos\theta_1)\cos 2m\tau$$
$$= \frac{(2n+2m-1)!}{2\pi}\sum_{p=0}^{n-1}\left\{\frac{(-1)^p}{(2p)!(2n-2p-1)!}(\cos\theta)^{2p}(\sin\theta)^{2n-2p-1}\right.$$
$$\times\int_{-\pi}^{\pi}[(\cos\beta+i\sin\beta\cos\varphi)^{2n-2p-1}$$
$$\left.+(\cos\beta-i\sin\beta\cos\varphi)^{2n-2p-1}](\sin\varphi)^{2p}\cos 2m\varphi\, d\varphi\right\}.$$

Introducing this in (46), and taking θ as integration variable instead of x we find with the aid of (38)

$$h_{mn}(\varkappa) = B_{mn}(\cos\beta)^{2n-1}\sum_{p=0}^{n-1}X_{mnp}(\beta)\,\psi_{np}(\gamma), \tag{48}$$

where

$$X_{mnp}(\beta) = \frac{(-1)^p(2n-2m-1)!}{\pi(2p)!(2n-2p-1)!}\int_{-\pi}^{\pi}[(\cos\beta+i\sin\beta\cos\varphi)^{2n-2p-1}$$
$$+(\cos\beta-i\sin\beta\cos\varphi)^{2n-2p-1}](\sin\varphi)^{2p}\cos 2m\varphi\, d\varphi. \tag{49}$$

We note some special cases of (49). When $p = 0$, we have

$$X_{mn0}(\beta) = \frac{(2n-2m-1)!}{\pi(2n-1)!}\left[\int_{-\pi}^{\pi}(\cos\beta+i\sin\beta\cos\varphi)^{2n-1}\cos 2m\varphi\, d\varphi\right.$$
$$\left.+\int_{-\pi}^{\pi}(\cos\beta-i\sin\beta\cos\varphi)^{2n-1}\cos 2m\varphi\, d\varphi\right],$$

and by (25)

$$\int_{-\pi}^{\pi}(\cos\beta+i\sin\beta\cos\varphi)^{2n-1}\cos 2m\varphi\, d\varphi = \frac{(-1)^m 2\pi(2n-1)!}{(2n+2m-1)!}P_{2n-1}^{2m}(\cos\beta),$$

which remains unchanged if we change the sign of i. Hence

$$X_{mn0}(\beta) = \frac{(-1)^m 4(2n-2m-1)!}{(2n+2m-1)!}P_{2n-1}^{2m}(\cos\beta). \tag{50}$$

Next, let $p = n-1$; then (49) gives

$$X_{m,n,n-1}(\beta) = \frac{(-1)^{n-1}(2n-2m-1)!}{\pi(2n-2)!}\int_{-\pi}^{\pi}2\cos\beta(\sin\varphi)^{2n-2}\cos 2m\varphi\, d\varphi,$$

and since the integrand has the period 2π, we may replace φ by $\varphi+\frac{1}{2}\pi$ without changing the limits of integration, so that

$$(51)\quad \int_{-\pi}^{\pi} (\sin\varphi)^{2n-2} \cos 2m\varphi \, d\varphi = (-1)^m \int_{-\pi}^{\pi} (\cos\varphi)^{2n-2} \cos 2m\varphi \, d\varphi,$$

whence, applying (31),

$$(52)\quad X_{m,n,n-1}(\beta) = \frac{(-1)^{n-m-1}(2n-2m-1)!}{2^{2n-4}(n-m-1)!\,(n+m-1)!} \cos\beta \ \text{ for } 0 \leq m \leq n-1.$$

Finally, for $\beta = 0$, (49) becomes

$$X_{mnp}(0) = \left(\frac{(-1)^p\,(2n-2m-1)!}{\pi\,(2p)!\,(2n-2p-1)!}\right) 2\int_{-\pi}^{\pi} (\sin\varphi)^{2p} \cos 2m\varphi \, d\varphi,$$

or by (51) and (31),

$$(53)\quad \begin{aligned} X_{mnp}(0) &= \frac{(-1)^{p-m}(2n-2m-1)!}{2^{2p-2}(p-m)!\,(p+m)!\,(2n-2p-1)!} \quad \text{for } p \geq m, \\ &= 0 \quad \text{for } p < m. \end{aligned}$$

In the general case, we may evaluate (49) by means of the well known formula

$$(\sin\varphi)^{2p} = \frac{1}{2^{2p}} \frac{(2p)!}{p!\,p!} + 2\sum_{s=1}^{p}\left(\frac{(-1)^s\,(2p)!}{(p-s)!\,(p+s)!} \cos 2s\varphi\right),$$

whence

$$(54)\quad \begin{aligned} (\sin\varphi)^{2p}\cos 2m\varphi = \frac{1}{2^{2p}}\Bigg[&\frac{(2p)!}{p!\,p!}\cos 2m\varphi \\ &+ \sum_{s=1}^{p}\left(\frac{(-1)^s\,(2p)!}{(p-s)!\,(p+s)!}\cos 2(m+s)\varphi\right) \\ &+ \sum_{s=1}^{p}\left(\frac{(-1)^s\,(2p)!}{(p-s)!\,(p+s)!}\cos 2(m-s)\varphi\right)\Bigg]. \end{aligned}$$

Introducing this in (49) and using (25), it is seen that $X_{mnp}(\beta)$ is expressible linearly in the various functions $P_{2n-2p-1}^{2s}(\cos\beta)$, where s takes all the values from 0 to $n-p-1$, the coefficients having a very simple form. For the small values of m, n and p which we need in the next paragraph, it is, however, quite as expedient to evaluate (49) directly, by expanding the binomials and applying (31), in the cases not covered by (50) and (52).

5. Evaluation of the coefficients up to $n = 3$ inclusive. From (15) it follows that $|R_3| \leq |\varphi_4(\alpha, x)|\lambda^8 + |R_4|$. Evaluating $\varphi_4(\alpha, x)$ by the method of the preceding paragraph, and computing its numerical value

for $\lambda = 0.35$ and various values of α and $\varkappa$, as well as estimating $|R_4|$ by means of (21), it is found that for $\lambda \leqq 0.35$, we have $|R_3| < 0.002$.

To obtain I/L with an error less than two units in the third decimal place, it is therefore sufficient to retain terms containing powers of λ not higher than the sixth, so that we only need to calculate $f_{mn}(\varkappa)$ for $n = 1$, 2 and 3. Writing

$$\cos\gamma = b, \tag{55}$$

we obtain from (39)

$$\psi_{10}(\gamma) = b,$$

$$\psi_{20}(\gamma) = b - \frac{2}{3} b^3 + \frac{1}{5} b^5,$$

$$\psi_{21}(\gamma) = \frac{1}{3} b^3 - \frac{1}{5} b^5,$$

$$\psi_{30}(\gamma) = b - \frac{4}{3} b^3 + \frac{6}{5} b^5 - \frac{4}{7} b^7 + \frac{1}{9} b^9,$$

$$\psi_{31}(\gamma) = \frac{1}{3} b^3 - \frac{3}{5} b^5 + \frac{3}{7} b^7 - \frac{1}{9} b^9,$$

$$\psi_{32}(\gamma) = \frac{1}{5} b^5 - \frac{2}{7} b^7 + \frac{1}{9} b^9.$$

The values of A_{mn} and B_{mn} computed from (33) and (44) are

$$\begin{array}{lll} A_{11} = 2, & & \\ A_{12} = 1, & A_{22} = 30, & \\ A_{13} = -21/4, & A_{23} = 175, & A_{33} = 1890, \end{array}$$

$$\begin{array}{lll} B_{01} = 2, & & \\ B_{02} = 1, & B_{12} = 10, & \\ B_{03} = -7/12, & B_{13} = 49, & B_{23} = 378. \end{array}$$

From (49)–(54) we obtain the following values of $X_{mnp}(\beta)$:

$$X_{010}(\beta) = 4\cos\beta,$$

$$X_{020}(\beta) = -6\cos\beta + 10\cos^3\beta,$$

$$X_{021}(\beta) = -6\cos\beta,$$

$$X_{120}(\beta) = -\frac{1}{2}\cos\beta(1-\cos^2\beta),$$

$$X_{121}(\beta) = \frac{1}{2}\cos\beta,$$

$$X_{030}(\beta) = \frac{15}{2}\cos\beta - 35\cos^3\beta + \frac{63}{2}\cos^5\beta,$$

$$X_{031}(\beta) = 15\cos\beta - 35\cos^3\beta,$$

$$X_{032}(\beta) = \frac{15}{2}\cos\beta,$$

$$X_{130}(\beta) = \left(\frac{1}{4}\cos\beta - \frac{3}{4}\cos^3\beta\right)(1-\cos^2\beta),$$

$$X_{131}(\beta) = \frac{1}{2}\cos^3\beta,$$

$$X_{132}(\beta) = -\frac{1}{4}\cos\beta,$$

$$X_{230}(\beta) = \frac{1}{2^5 3}\cos\beta(1-\cos^2\beta)^2,$$

$$X_{231}(\beta) = -\frac{1}{2^4}\cos\beta(1-\cos^2\beta),$$

$$X_{232}(\beta) = \frac{1}{2^5 3}\cos\beta.$$

The calculation is now readily completed by means of (40), (48) and (47), and writing

(56) $$\cos^2\beta = a,$$

we obtain the results stated in the introductory paragraph.

6. **The special case** $\alpha = 0$, $\varkappa = 0$. The result of this paragraph is needed for the computation of the table at the end of paragraph 3, and, since it is derived by an independent method, it will also serve as a partial check on the numerical values of the coefficient in the general case.

When $\alpha = 0$, it is obvious that $I' = I''$, so that $I = 2I'$, and making also $\varkappa = 0$, it follows from (2), (3) and (4) that

(57) $$\frac{I}{L} = F(\lambda) + F(-\lambda),$$

where

(58) $$F(\lambda) = \int_{-1}^{1}\int_{-1}^{1}\frac{2\lambda}{\varrho}\,dx\,dy,$$

and

(59) $$\varrho^2 = (1-\lambda)^2 + (x-\lambda y)^2.$$

To evalute (58), we follow Campbell (loc. cit.) in introducing the identity

$$\frac{\lambda}{\varrho} = \frac{\partial}{\partial x}\frac{\lambda x}{\varrho} + \frac{\partial}{\partial y}\frac{\lambda y}{\varrho} + \frac{\partial^2 \varrho}{\partial x\, \partial y},$$

from which it is seen that

$$\begin{aligned} F(\lambda) = & \int_{-1}^{1}\frac{2\lambda\, dy}{[(1-\lambda)^2+(1-\lambda y)^2]^{1/2}} + \int_{-1}^{1}\frac{2\lambda\, dy}{[(1-\lambda)^2+(1+\lambda y)^2]^{1/2}} \\ & + \int_{-1}^{1}\frac{2\lambda\, dx}{[(1-\lambda)^2+(x-\lambda)^2]^{1/2}} + \int_{-1}^{1}\frac{2\lambda\, dx}{[(1-\lambda)^2+(x+\lambda)^2]^{1/2}} \\ & + 4\sqrt{2}\,(1-\lambda) - 4\sqrt{2\,(1+\lambda^2)}, \end{aligned}$$

or evaluating the integrals,

$$\begin{aligned} F(\lambda) = & \; 4(1+\lambda)\log\left(\frac{1+\lambda+\sqrt{2(1+\lambda^2)}}{(\sqrt{2}+1)(1-\lambda)}\right) + 8\lambda\log(\sqrt{2}+1) \\ & + 4\sqrt{2}(1-\lambda) - 4\sqrt{2(1+\lambda^2)}. \end{aligned} \tag{60}$$

To expand I/L in powers of λ, we note that

$$\begin{aligned} F(\lambda) = & \; 4\sqrt{2}(1+\lambda)\int_0^{\lambda}\frac{d\lambda}{(1-\lambda)(1+\lambda^2)^{1/2}} + 8\lambda\log(\sqrt{2}+1) \\ & + 4\sqrt{2}(1-\lambda) - 4\sqrt{2(1+\lambda^2)}, \end{aligned} \tag{61}$$

as is readily seen by differentiating the logarithm occurring in (60). It now also follows that

$$\begin{aligned} F(-\lambda) = & -4\sqrt{2}(1-\lambda)\int_0^{\lambda}\frac{d\lambda}{(1+\lambda)(1+\lambda^2)^{1/2}} - 8\lambda\log(\sqrt{2}+1) \\ & + 4\sqrt{2}(1+\lambda) - 4\sqrt{2(1+\lambda^2)}, \end{aligned}$$

and adding the two expressions,

$$\begin{aligned} \frac{I}{L} = 8\sqrt{2}\Bigg[& \int_0^{\lambda}\frac{\lambda\, d\lambda}{(1-\lambda^2)(1+\lambda^2)^{1/2}} \\ & + \lambda\int_0^{\lambda}\frac{d\lambda}{(1-\lambda^2)(1+\lambda^2)^{1/2}} + 1 - (1+\lambda^2)^{1/2}\Bigg]. \end{aligned} \tag{62}$$

Introducing the binomial expansions of $(1-\lambda^2)^{-1}$, $(1+\lambda^2)^{-1/2}$ and $(1+\lambda^2)^{1/2}$, we obtain the desired expansion

$$\frac{I}{L} = \sum_{n=1}^{\infty} c_n \lambda^{2n}, \tag{63}$$

where

$$\begin{aligned} c_1 &= 8\sqrt{2}, \\ c_n &= 4\sqrt{2}\left[\left(\frac{1}{n}+\frac{2}{2n-1}\right)\left(1+\sum_{s=1}^{n-1}\frac{(-1)^s\cdot 1\cdot 3\cdot 5\cdots(2s-1)}{2^s\cdot s!}\right)\right. \\ &\quad \left. + \frac{(-1)^n\cdot 1\cdot 3\cdot 5\cdots(2n-3)}{2^{n-1}\cdot n!}\right], \text{ for } n>1. \end{aligned} \tag{64}$$

Computing a few of these coefficients, we find

$$\begin{aligned} \frac{I}{L} = \sqrt{2}\Big(8\lambda^2 &+ \frac{10}{3}\lambda^4 + \frac{31}{3\cdot 5}\lambda^6 + \frac{85}{2^3\cdot 7}\lambda^8 + \frac{859}{2^4\cdot 3^2\cdot 5}\lambda^{10} + \frac{771}{2^6\cdot 11}\lambda^{12} \\ &+ \frac{10851}{2^7\cdot 7\cdot 13}\lambda^{14} + \frac{3249}{2^{10}\cdot 5}\lambda^{16} + \frac{150065}{2^{12}\cdot 3\cdot 17}\lambda^{18} + \cdots\Big). \end{aligned} \tag{65}$$

NEW YORK, N. Y.

ON THE DEVELOPMENT OF CONTINUOUS FUNCTIONS IN SERIES OF TCHEBYCHEFF POLYNOMIALS*

BY

J. A. SHOHAT (JACQUES CHOKHATE)

Introduction. Consider a system of orthogonal and normal Tchebycheff polynomials

$$\varphi_n(x) = a_n x^n + \cdots \qquad (n = 0, 1, 2, \cdots;\ a_n > 0)$$

corresponding to a certain interval (a, b) with the characteristic function $p(x)$ integrable and not negative on (a, b). Thus we have

$$\int_a^b p(x)\,\varphi_m(x)\,\varphi_n(x)\,dx = \begin{cases} 0, & m \neq n, \\ 1, & m = n. \end{cases}$$

For any function $f(x)$ we have a formal development as follows:

$$\text{(I)} \qquad f(x) \sim \sum_0^\infty A_i\,\varphi_i(x), \qquad A_i = \int_a^b p(x) f(x)\,\varphi_i(x)\,dx,$$

provided, of course, the right hand integrals exist.

Let us write

$$\text{(II)} \qquad f(x) = \sum_0^n A_i\,\varphi_i(x) + r_{n,f}(x) \equiv P_{n,f}(x) + r_{n,f}(x).$$

The question arises as to the convergence of the development (I) to $f(x)$ or—what is the same—the behavior of $r_{n,f}(x)$ in (II) for $n \to \infty$.

The case $(a, b) = (-1, 1)$, $p(x) = 1$ leads to Legendre's polynomials; it has been treated by Professor D. Jackson.†

In this paper we follow the method given by Professor Jackson in order to investigate the convergence of the development (I) involving *Tchebycheff's polynomials in general.* Hereafter, *the interval* (a, b) *is supposed to be finite, and* $f(x)$ *to be continuous on* (a, b).

1. $r_{n,f}(x)$ **expressed as a definite integral.** We obtain easily, using the formulas for A_i,

$$r_{n,f}(x) = f(x) - \int_a^b p(y) f(y) \sum_0^n \varphi_i(x)\,\varphi_i(y)\,dy,$$

* Presented to the Society, April 19, 1924.

† D. Jackson, *On the degree of convergence of a continuous function according to Legendre's polynomials*, these Transactions, vol. 13 (1912), pp. 305–318.

which gives, for $f(x) \equiv 1$,

$$1 = \int_a^b p(y) \sum_0^n \varphi_i(x)\varphi_i(y)\,dy.$$

Hence,

$$r_{n,f}(x) = \int_a^b p(y)\,[f(x) - f(y)]\,K_n(x, y)\,dy,$$

$$K_n(x, y) = \sum_0^n \varphi_i(x)\varphi_i(y);$$

$$r_{n,f}(x) = \frac{a_n}{a_{n+1}} \int_a^b p(y)\,[f(x) - f(y)]\,\frac{\varphi_{n+1}(x)\varphi_n(y) - \varphi_n(x)\varphi_{n+1}(y)}{x - y}\,dy,$$

since

$$K_n(x, y) = \frac{a_n}{a_{n+1}}\,\frac{\varphi_{n+1}(x)\varphi_n(y) - \varphi_n(x)\varphi_{n+1}(y)}{x - y}.*$$

Following Jackson's method we get for any polynomial $Q_n(x)$, of degree $\leqq n$,

$$0 = \int_a^b p(y)\,[Q_n(x) - Q_n(y)]\,K_n(x, y)\,dy,$$

$$r_{n,f}(x) = \int_a^b p(y)\,[\varphi(x) - \varphi(y)]\,K_n(x, y)\,dy,$$

$$\varphi(x) \equiv f(x) - Q_n(x).$$

We substitute here for $Q_n(x)$ a special polynomial, namely *the polynomial* $T_{n,f}(x)$ *of best approximation to* $f(x)$ *on* (a, b) (of degree n). Thus we get two formulas for $r_{n,f}(x)$:

(1) $$r_{n,f}(x) = \int_a^b p(y)\,[\varphi(x) - \varphi(y)]\,K_n(x, y)\,dy,$$

$$K_n(x, y) = \sum_0^n \varphi_i(x)\varphi_i(y); \quad \varphi(x) \equiv f(x) - T_{n,f}(x);$$

(2) $$r_{n,f}(x) = \frac{a_n}{a_{n+1}} \int_a^b p(y)\,[\varphi(x) - \varphi(y)]\,\frac{\varphi_{n+1}(x)\varphi_n(y) - \varphi_n(x)\varphi_{n+1}(y)}{x - y}\,dy.$$

Denote by $E_n(f)$ the *best approximation* on (a, b) of $f(x)$ by means of a polynomial of degree n, i. e.

(3) $$E_n(f) = \max |f(x) - T_{n,f}(x)| \text{ for } a \leqq x \leqq b.$$

---

* Darboux, *Mémoire sur l'approximation des fonctions de très grands nombres*, Journal de Mathématiques Pures et Appliquées, ser. 3, vol. 4 (1878), pp. 5–60, 377–416; p. 413.

Using Schwarz's inequality, we get from (1)

$$|r_{n,f}(x)| \leqq 2\,E_n(f)\sqrt{\int_a^b p(y)\,dy}\,\sqrt{\int_a^b p(y)\,K_n^2(x,y)\,dy}$$
$$= 2\,E_n(f)\sqrt{\int_a^b p(y)\,dy}\,\sqrt{\sum_0^n \varphi_i^2(x)},$$

$$|r_{n,f}(x)| \leqq 2\,Q\,E_n(f)\sqrt{K_n(x)} \qquad \left(Q^2 = \int_a^b p(y)\,dy\right), \tag{4}$$

$$K_n(x) \equiv K_n(x,x) = \sum_0^n \varphi_i^2(x). \tag{5}$$

$E_n(f)$, as a function of n, has been investigated by Lebesgue, de la Vallée-Poussin, S. Bernstein, W. Stekloff, and in particular by D. Jackson.*

Table A: $E_n(f)$

Conditions imposed on $f(x)$	$E_n(f) =$	Author
1. $\|f(x_2)-f(x_1)\| \leqq \omega(\delta)$ for $\|x_2-x_1\| \leqq \delta$ $(a \leqq x_1,\ x_2 \leqq b)$.	$O\left(\omega\left(\frac{b-a}{n}\right)\right)$	D. Jackson
2. $\|f^{(p)}(x_2)-f^{(p)}(x_1)\| < \lambda\,\|x_2-x_1\|^\alpha$ (Lipschitz condition of order α $(a \leqq x_1,\ x_2 \leqq b,\ \lambda = \text{const.};$ $f^{(0)}(x) \equiv f(x))$).	$O\left(\frac{1}{n^{p+\alpha}}\right)$	"
3. $f^{(p)}(x)$ is continuous on (a, b).	$o\left(\frac{1}{n^p}\right)$	"
4. $\|f(x+\delta)-f(x)\|\cdot\|\log\delta\| < \lambda (=\text{const.})$	$O\left(\frac{1}{\log n}\right)$	"
5. $\|f(x+\delta)-f(x)\|\cdot\|\log\delta\| \to 0,\ \delta \to 0$ (Dini-Lipschitz condition).	$o\left(\frac{1}{\log n}\right)$	Lebesgue
6. $0 < N < f^{(n)}(x) < M$ for $a \leqq x \leqq b$.	$\frac{2N}{n!} < \left(\frac{4}{b-a}\right)^n E_n(f) < \frac{2M}{n!}$	S. Bernstein; W. Stekloff
7. $f^{(p)}(x)$ exists for every p.	$n^p E_n(f) \to 0,\ n \to \infty,$ for every p	S. Bernstein

We see from Table A that in order to evaluate $r_{n,f}(x)$ by means of (1, 2, 4), we need to know the order of $\varphi_n(x)$ or $K_n(x)$ with respect to n.

* D. Jackson, *Über die Genauigkeit der Annäherung stetiger Funktionen*, Dissertation, Göttingen, 1911, pp. 1–96.

2. **Order of $K_n(x)$ (with respect to n).** It can be proved easily that

$$(6)\qquad \frac{1}{K_n(z)} = \min \int_a^b p(y)[1+Z_1(y-z)+\cdots+Z_n(y-z)^n]^2\,dy^*.$$

Therefore, using the notation $K_n(p;z)$, we conclude that

$$(7)\qquad \begin{gathered} p_2(x) \geqq p(x) \geqq p_1(x) \text{ for } a \leqq x \leqq b \text{ implies} \\ K_n(p_2;z) \leqq K_n(p;z) \leqq K_n(p_1;z). \end{gathered}$$

On the other hand, to the characteristic function

$$(8)\qquad \begin{gathered} p_1(x) = (x-a)^{\alpha-1}(b-x)^{\beta-1}\Pi(x) \qquad (\alpha,\beta>0), \\ \Pi(x) \text{ a polynomial } (\Pi(a)\Pi(b) \neq 0), \end{gathered}$$

there corresponds a special system of Tchebycheff's polynomials, a generalization of Jacobi's polynomials ($\Pi(x)\equiv 1$), and I have obtained the asymptotic expression for $K_n(p_1;z)$ at any point z in (a,b)†.

Thus we have

$$(9)\qquad \begin{gathered} K_n(p_1;a) \sim n^{2\alpha}, \qquad K_n(p_1;b) \sim n^{2\beta}, \\ K_n(p_1;z) \sim n^{2m+1}, \end{gathered}$$

z being a root of $\Pi(x)$ of multiplicity $2m \geqq 0$ $(a+\epsilon \leqq z \leqq b-\epsilon)$‡. These results enable us to prove the following

THEOREM I. (i) *Suppose the point $x = z$ be inside the interval (a, b): $a+\epsilon \leqq z \leqq b-\epsilon$, and that there exist finite numbers $k>-1$, $A>0$, c, d such that*

$$\frac{p(x)}{|x-z|^k} \geqq A \text{ for } (a \leqq)\ c \leqq x \leqq d\ (\leqq b) \qquad (c<z<d).$$

Let us take the smallest k possible satisfying the above conditions and $k=0$ in case $p(z)>0$. Then $K_n(p;z) = O(n^{2k'+1})$, where k' is the smallest integer $\geqq k/2$. In particular $K_n(p;z) = O(n)$ for $k \leqq 0$.

(ii) *Suppose the point $x = z$ coincides with one of the end points of (a,b), say $z = a$. If*

$$\frac{p(x)}{|x-a|^k} \geqq A \qquad (k>-1,\ A>0;\ a \leqq x \leqq c\ (\leqq b)),$$

---

* See my paper (where the proof is given for $z = 0$), Jacques Chokhate, *Sur le développement de l'intégrale $\int_a^b [p(y)/(x-y)]\,dy$ en fraction continue et sur les polynomes de Tchebycheff*, Rendiconti del Circolo Matematico di Palermo, vol. 47 (1923), pp. 25–46; p. 41.

† *On the asymptotitc properties of a certain class of Tchebycheff's polynomials*, read before the International Mathematical Congress, Toronto, August, 1924.

‡ Hereafter ϵ stands for an arbitrarily small but fixed quantity.

then

$$K_n\ (p; a) = O(n^{2k+2}).$$

Proof. (i) Consider the characteristic function corresponding to the interval (c, d) and defined as follows:

$$p_1(x) = A'(x-z)^{2k'} \text{ in } (c,\ d) \qquad (A'>0;\ c<z<d).$$

We have, then, A' being sufficiently small,

$$p(x) \geqq p_1(x) \text{ for } c \leqq x \leqq d.$$

Therefore (see 7,9), since

$$\min \int_a^b p(y)[1+Z_1(y-z)+\cdots+Z_n(y-z)^n]^2\,dy$$
$$\geqq \min \int_c^d p(y)\,[1+Z_1(y-z)+\cdots+Z_n(y-z)^n]^2\,dy,$$
$$K_n(p; z) \leqq K_n(p_1; z) = O(n^{2k'+1}), \qquad \text{Q. E. D.}$$

In a similar, slightly modified, way we prove the statement (ii) of our theorem. Formula (4) leads to the following

COROLLARY. *If $p(x)$ satisfies the conditions of Theorem* I, *then*

$$\text{(i)} \qquad |r_{n,f}(z)| < \tau E_n(f)\, n^{k'+1/2} \qquad (a+\varepsilon \leqq z \leqq b-\varepsilon),$$

$$\text{(ii)} \qquad |r_{n,f}(z)| < \tau E_n(f)\, n^{k+1} \qquad ((z-a)(z-b) = 0).^*$$

3. Order of $r_{n,f}(x)$ (with respect to n) (method of D. Jackson). For any system of Tchebycheff's polynomials the following inequality holds:

$$\frac{a_n}{a_{n+1}} < \frac{b-a}{2}.\dagger \tag{10}$$

Consider two cases:

Case I. *The point $x = z$ is inside the interval (a, b)*: $a<c<z<d<b$.

We write (1, 2) as follows:

$$r_{n,f}(z) = \int_a^{c+\varepsilon} + \int_{c+\varepsilon}^{z-\varepsilon_n} + \int_{z-\varepsilon_n}^{z+\varepsilon_n} + \int_{z+\varepsilon_n}^{d-\varepsilon} + \int_{d-\varepsilon}^{b} = i_1+i_2+i_3+i_4+i_5,$$
$$\varepsilon_n > 0, \quad \varepsilon_n \to 0 \text{ for } n\to\infty. \tag{11}$$

---

* Hereafter we use τ to denote generally a fixed positive quantity, different, of course in different formulas, which does not depend on n.

† J. Chokhate, loc. cit., p. 33.

Here ε denotes a certain fixed positive quantity; n is supposed to be so large and ε so small that we have

$$\varepsilon_n < \varepsilon, \quad c + 2\varepsilon \leqq z \leqq d - 2\varepsilon.$$

Suppose the system of Tchebycheff's polynomials under consideration subjected to following conditions:

$$\left.\begin{array}{lll}(12) & p(x) < P \ (= \text{fixed const.}) & \\ (13) & |\varphi_n(x)| < \tau n^{\sigma} \quad (n = 1, 2, \cdots; \ \sigma > -\tfrac{1}{2}{}^*) & \end{array}\right\} (c \leqq x \leqq d),$$

where P, τ, σ do not depend upon x, nor upon n.

Consider first i_1 and i_5 in (11). Here we use formula (2), since $1/|x-y| < \varepsilon$. Using (3, 10) we get

$$|i_1|, |i_5| < \frac{b-a}{2} E_n(f) \left\{ |\varphi_{n+1}(z)| \int_a^b p(y) |\varphi_n(y)| \, dy + |\varphi_n(z)| \int_a^b p(y) |\varphi_{n+1}(y)| \, dy \right\};$$

$$(14) \qquad |i_1|, |i_5| < \tau E_n(f) n^{\sigma},$$

assuming only the condition $|\varphi_n(z)| < \tau n^{\sigma}$ (since

$$\int_a^b p(y) |\varphi_i(y)| \, dy < \sqrt{\int_a^b p(y) \, dy}$$

by Schwarz's inequality). We use the same formula (2) to estimate i_2 and i_4.

Putting $z - y = u$, we get

$$|i_2|, |i_4| < \tau n^{2\sigma} E_n(f) \int_{\varepsilon_n}^{b-a} \frac{du}{u};$$

$$(15) \qquad |i_2|, |i_4| < \tau n^{2\sigma} E_n(f) |\log \varepsilon_n|,$$

under conditions (12, 13). In order to estimate i_3 in (11), we write

$$(16) \qquad i_3 = \int_{z-\varepsilon_n}^{z+\varepsilon_n} p(y) [\varphi(z) - \varphi(y)] \sum_0^n \varphi_i(z) \varphi_i(y) \, dy,$$

which gives

$$|i_3| < \tau E_n(f) \left[\varphi_0^2 + \sum_1^n i^{2\sigma} \right] \int_{z-\varepsilon_n}^{z+\varepsilon_n} p(y) \, dy;$$

$$(17) \qquad |i_3| < \tau E_n(f) n^{2\sigma+1} \varepsilon_n,$$

under conditions (12,13). If we replace condition (13) by a less restrictive one,

* $\sigma \leqq -\frac{1}{2}$ does not occur in applications (see below, p. 544).

$$|\varphi_n(z)| < \tau n^\sigma \qquad (n = 1, 2, \cdots; \sigma > -\tfrac{1}{2}), \tag{18}$$

we can estimate i_2, i_4 and i_3 as follows:

Apply Schwarz's inequality to i_2, i_4 in (11). We get, since here $1/|z-y| \leqq 1/\varepsilon_n$,

$$|i_2|,\ |i_4| < \tau \frac{E_n(f)}{\varepsilon_n} n^\sigma \tag{19}$$

assuming only the condition (18). Assuming two conditions, (12) and (18), we get

$$|i_2| < \tau E_n(f) \sqrt{\int_{a+\varepsilon}^{z-\varepsilon_n} \frac{dy}{(z-y)^2}}\ \sqrt{\varphi_{n+1}^2(z) + \varphi_n^2(z)}\,;^*$$

$$|i_2|,\ |i_4| < \tau \frac{E_n(f)}{\sqrt{\varepsilon_n}} n^\sigma. \tag{20}$$

Similarly, applying Schwarz's inequality to i_3 in (16), we get

$$|i_3| < \tau E_n(f) \sqrt{\int_{z-\varepsilon_n}^{z+\varepsilon_n} p(y)\,dy}\ \sqrt{\int_a^b p(y) K_n^2(z, y)\,dy}\,;$$

$$|i_3| < \tau E_n(f) \sqrt{\varepsilon_n}\ \sqrt{K_n(z)} \quad \text{(under conditions (12));} \tag{21}$$

$$|i_3| < \tau E_n(f) \sqrt{\varepsilon_n}\, n^{\sigma+1} \qquad \text{(under conditions (12, 18)).} \tag{22}$$

Case II. $(z-a)(z-b) = 0$; say $z = b$. Assume, as above,

$$\left.\begin{array}{ll} (23) & p(x) < P \\ (24) & |\varphi_n(x)| < \tau n^\sigma \quad (n = 1, 2, \cdots; \sigma > -\tfrac{1}{2}) \end{array}\right\} ((a \leqq)\, c \leqq x \leqq b),$$

or

$$|\varphi_n(b)| < \tau n^\sigma \quad (n = 1, 2, \cdots; \sigma > -\tfrac{1}{2}). \tag{25}$$

Write (1,2) as follows:

$$r_{n,f}(b) = \int_a^{c+\varepsilon} + \int_{c+\varepsilon}^{b-\varepsilon_n} + \int_{b-\varepsilon_n}^{b} = i_1 + i_2 + i_3,$$

$$\varepsilon_n > 0, \quad \varepsilon_n \to 0 \text{ for } n \to \infty; \quad \varepsilon > 0; \quad \varepsilon_n < \varepsilon; \quad c + z\varepsilon < b.$$

Following the preceding discussion we estimate i_1, i_2, i_3 in a manner quite similar to that given above and find similar results.

We proceed to specify the infinitesimal ε_n. Take $\varepsilon_n = n^{-\beta}$, with $\beta > 0$, and choose β so as to make $r_{n,f}(z)$ of the highest order possible with respect to $1/n$. The results thus found (using the expressions above for i_1, i_2, i_3, i_4, i_5) are summarized in the following table.

---

* Similar inequality for $|i_4|$.

Table B: $r_{n,f}(x)$

Case	Conditions imposed on $p(x)$, $\varphi_n(x)$			$\lvert r_{n,f}(z)\rvert < \tau E_n(f) h_n$* with $h_n =$
	$p(x)$ is	$\varphi_n(x) < \tau n^\sigma$ for	with σ	
1. $a<c<z<d<b$	bounded for $c \leq x \leq d$	$c \leq x \leq d$	< 0	n^σ (impossible; see below)
2. "	"	"	$0 \leq \sigma < \frac{1}{4}$	$n^{2\sigma} \log n$
3. "	"	"	$\sigma \geq \frac{1}{4}$	$n^{\sigma+1/4}$
4. "	"	at the point $x = z$	$\sigma > -\frac{1}{2}$	$n^{\sigma+1/4}$
5. Same results hold in case $(z-a)(z-b)=0$ under analogous conditions imposed on $p(x)$ and $\varphi_n(x)$.				
6.	no conditions	at the point $x = z$	$\sigma > -\frac{1}{2}$	$n^{\sigma+1/2}$
7. any	no conditions			$\sqrt{K_n(z)}$†

The most interesting case is

$$(26) \qquad \sigma = 0; \quad |r_{n,f}(z)| < \begin{cases} \tau E_n(f) \log n & \text{(under conditions (12, 13))}, \\ \tau E_n(f)\, n^{1/4} & \text{(under conditions (12, 18))}, \\ \tau E_n(f)\, n^{1/2} & \text{(under condition (18))}. \end{cases}$$

The condition (18) with $\sigma = 0$ holds, for instance, in the case of the characteristic function (8) (see second footnote on page 540) at any point $x = z$ $[(z-a)(z-b) \neq 0]$, where $\Pi(z) \neq 0$.

Another case, where we have (18) satisfied with $\sigma = 0$, is given by G. Szegö.‡

It remains to prove that *it is impossible to have* (12, 13) *with* $\sigma < 0$ (see Table B, case 1).

In fact, the contrary assumption gives

$$(27) \quad \frac{r_{n,f}(x)}{n^\sigma} \to 0 \quad \text{for} \quad n \to \infty \quad \text{uniformly} \qquad (\sigma < 0;\ c < c' \leq x \leq d' < d),$$

since $E_n(f) \to 0$ with $1/n$ for every continuous function.

Writing in general $E_n(f; a, b)$, we get, from the very definition of this quantity,

$$E_n(f; c', d') \leq \max |r_{n,f}(x)| \quad \text{for} \quad c' \leq x \leq d'.$$

* τ is a fixed constant, not depending on n, nor on z (see (12, 13)).

† In some cases we know the order of $K_n(z)$, but not that of $\varphi_n(z)$.

‡ G. Szegö, *Über den asymptotischen Ausdruck von Polynomen*, Mathematische Annalen, vol. 86 (1922), pp. 114–140; p. 139.

Therefore, according to (27),

$$E_n(f; c', d') = o(n^\sigma) \quad \text{with} \quad \sigma < 0,$$

for every continuous function, which is impossible, because, as was established by D. Jackson,* for any $\sigma < 0$ there always exists a continuous function $f(x)$, for which

$$\frac{E_n(f; c', d')}{n^\sigma} \to 0 \quad \text{for} \quad n \to \infty, \quad \text{Q. E. D.†}$$

4. **Order of $\varphi_n(x)$ (with respect to n).‡** We modify slightly our notations for Tchebycheff's polynomials and write in general, the characteristic function being $p(x)$,

$$(28) \qquad \varphi_n(p; x) = a_n(p) x^n + \cdots \qquad (n = 0, 1, 2, \cdots; a_n(p) > 0).$$

We shall proceed to compare $\varphi_n(p; x)$ and $\varphi_n(q; x)$ corresponding to the same interval (a, b).

For this purpose consider

$$(29) \qquad \begin{aligned} \Delta_{p,q} &= \int_a^b q(x)\,[\varphi_n(q; x) - \varphi_n(p; x)]^2\, dx = i_1 - 2i_2 + i_3, \\ i_1 &= \int_a^b q(x)\,\varphi_n^2(q; x)\, dx = 1, \\ i_2 &= \int_a^b q(x)\,\varphi_n(p; x)\,\varphi_n(q; x)\, dx = \frac{a_n(p)}{a_n(q)}, \\ i_3 &= \int_a^b q(x)\,\varphi_n^2(p; x)\, dx = 1 + \int_a^b [q(x) - p(x)]\,\varphi_n^2(p; x)\, dx \\ &= 1 + \theta_1 \left(\frac{q-p}{p}\right)_{\max} \qquad (|\theta_1| \leqq 1), \end{aligned}$$

where in general $(u)_{\max}$, $(u)_{\min}$ stand for the least upper and greatest lower bound respectively (or maximum and minimum) of $|u(x)|$ in (a, b).

We make use now of following inequalities:

$$\left(\frac{q}{p}\right)_{\min} < \frac{a_n^2(p)}{a_n^2(q)} < \left(\frac{q}{p}\right)_{\max}, \S$$

* Loc. cit., p. 56.

† But we may have $|\varphi_n(x)| < \tau n^\sigma$ with $\sigma < 0$ *at a certain point* $x = z$; e. g., for the polynomials of Jacobi (p. 540), with $\alpha, \beta < \frac{1}{2}$ at $x = a, b$ ($\sigma = \alpha - \frac{1}{2}, \beta - \frac{1}{2}$, respectively).

‡ The results of this paragraph are summarized in my article *Sur les polynomes de Tchebycheff*, Comptes Rendus, vol. 178 (1924), p. 2229. Here they are somewhat generalized.

§ J. Chokhate, *Sur quelques propriétés des polynomes de Tchebycheff*, Comptes Rendus, vol. 166 (1918), pp. 28-30.

which give

$$\left| \frac{a_n(p)}{a_n(q)} - 1 \right| < \left(\frac{q-p}{p} \right)_{\max}. \tag{30}$$

Thus we have, using (29),

$$i_2 = 1 + \theta_2 \left(\frac{q-p}{p} \right)_{\max} \qquad (|\theta_2| \leqq 1),$$

$$\Delta_{p,q} = \delta_n \left(\frac{q-p}{p} \right)_{\max} \qquad (0 < \delta_n < 3). \tag{31}$$

On the other hand, $Q_n(x)$ being an arbitrary polynomial of degree $\leqq n$, we have

$$Q_n(x) = \sum_0^n A_i \varphi_i(q;x), \qquad A_i = \int_a^b q(x)\, Q_n(x)\, \varphi_i(q;x)\, dx,$$

$$\begin{aligned} |Q_n(x)| &\leqq \sqrt{\sum_0^n A_i^2}\, \sqrt{\sum_0^n \varphi_i^2(q;x)}; \\ |Q_n(x)| &\leqq \sqrt{\int_a^b q(x)\, Q_n^2(x)\, dx}\, \sqrt{K_n(q;x)}, \quad K_n(q;x) \equiv \sum_0^n \varphi_i^2(q;x). \end{aligned} \tag{32}$$

Apply (32) to the polynomial $\varphi_n(p;x) - \varphi_n(q;x)$ and use (31). We get

$$\left.\begin{aligned} |\varphi_n(p,x) - \varphi_n(q;x)| &< \tau \sqrt{\left(\frac{q-p}{p} \right)_{\max}}\, \sqrt{K_n(q;x)} \\ |\varphi_n(p;x) - \varphi_n(q;x)| &< \tau \sqrt{\left(\frac{q-p}{q} \right)_{\max}}\, \sqrt{K_n(p;x)} \end{aligned}\right\} \quad (0 < \tau < \sqrt{3}). \tag{33}$$

Formulas (30, 33) lead to following

THEOREM II. *Suppose that $q(x)$, containing a parameter α, tends for $\alpha \to \alpha_0$ to $p(x)$ uniformly in (a, b), and that $p(x) \geqq p_{\min} > 0$ in (a, b). Then $\varphi_n(q;x) \to \varphi_n(p;x)$ uniformly in (a, b), and $a_n(q)$ under the above conditions tends uniformly (with respect to n) to $a_n(p)$.*

Proof. ε being chosen as small as we please, take $|\alpha - \alpha_0|$ sufficiently small in order to give, for $a \leqq x \leqq b$,

$$|q(x) - p(x)| < \frac{p_{\min}}{2},$$

$$|q(x) - p(x)| < \frac{\varepsilon^2 p_{\min}}{6 K_n},$$

$$|q(x) - p(x)| < \varepsilon p_{\min}, \qquad \text{for } a \leqq x \leqq b,$$

where $K_n = \max K_n(p; x)$ in (a, b). Then

$$q_{\min} > \frac{p_{\min}}{2}, \qquad \left(\frac{q-p}{p}\right)_{\max} < \frac{2(q-p)_{\max}}{p_{\min}},$$

$$|\varphi_i(p; x) - \varphi_i(q; x)| < \varepsilon \qquad (i = 0, 1, \cdots, n;\ a \leqq x \leqq b),$$

$$\left|\frac{a_n(p)}{a_n(q)} - 1\right| < \varepsilon \text{ for every } n, \qquad \text{Q. E. D.}$$

Consider first two special systems of Tchebycheff's polynomials:

$$\text{(34)}\quad \begin{array}{l} \varphi_n(p; x) \text{ with } p(x) = (x-a)^{\alpha-1}(b-x)^{\beta-1}\Pi(x), \\ \qquad (\alpha, \beta > 0;\ \Pi(a)\Pi(b) \neq 0);\ \Pi(x) \text{ a polynomial of degree } s; \end{array}$$

$$\text{(35)}\quad \begin{array}{l} \varphi_n(q; x) \text{ with } q(x) = (x-a)^{\alpha-1}(b-x)^{\beta-1}, \\ \qquad\qquad (\alpha, \beta > 0) \text{ (polynomials of Jacobi).} \end{array}$$

We have used these polynomials above (see pages 540, 544). We are now interested in finding what are the relations *between* $\varphi_n(p; x)$, $K_n(p; x)$ *and the degree* s *of* $\Pi(x)$ *in* (34).

For this purpose consider the development

$$\text{(36)}\quad \Pi(x)\varphi_n(p; x) = \sum_0^{n+s} A_i \varphi_i(q; x), \qquad A_i = \int_a^b p(x)\varphi_n(p; x)\varphi_i(q; x)\,dx,$$

where, as we see immediately,

$$\text{(37)}\qquad A_0 = A_1 = \cdots = A_{n-1} = 0.$$

On the other hand, as is well known,

$$\text{(38)}\quad \begin{array}{ll} |\varphi_n(q; z)| < \tau & (a+\varepsilon \leqq z \leqq b-\varepsilon), \\ |\varphi_n(q; a)| < \tau n^{\alpha-1/2}, & |\varphi_n(q; b)| < \tau n^{\beta-1/2}, \end{array}$$

where τ does not depend on z, nor on n.

Hence, (36, 37) give

$$\Pi(x)|\varphi_n(p; x)| \leqq \sqrt{\int_a^b p(x)\Pi(x)\varphi_n^2(p; x)\,dx}\ \sqrt{\sum_n^{n+s}\varphi_i^2(q; x)}$$

$$\leqq \sqrt{\Pi_{\max}}\ \sqrt{\sum_n^{n+s}\varphi_i^2(q; x)}.$$

Using (38), we get, n being sufficiently large (since $\varphi_n(q; x)$ does not depend on s):

$$\Pi(z)\,|\varphi_n(p;z)| < \tau\sqrt{s} \qquad (a+\varepsilon \leqq z \leqq b-\varepsilon), \tag{39}$$

$$\begin{gathered} |\varphi_n(p;z)| < \tau\sqrt{s} \qquad ((a<)\, c+\varepsilon \leqq z \leqq d-\varepsilon\,(<b);\ \Pi(x) \neq 0 \text{ in } (c,d)), \\ |\varphi_n(p;a)| < \tau\sqrt{s}\, n^{\alpha-1/2}; \qquad |\varphi_n(p;b)| < \tau\sqrt{s}\, n^{\beta-1/2}, \end{gathered} \tag{40}$$

where τ does not depend on z, nor on n, nor on s.

Formulas (40) answer the question stated above.

We return now to the general case. Assume that there exists a certain interval (c, d) such that

(41) *$p(x)$ is continuous and positive for $c \leqq x \leqq d$ $(a \leqq c;\ d \leqq b)$.*

Consider the polynomial $T_{m,p}(x)$ of best approximation to $p(x)$ in (c, d) of sufficiently large degree m. The polynomial $T_{m,p}(x)$ is also positive in (c, d). Now introduce the functions $\varphi_n(q; x)$, with

$$\begin{aligned} q(x) &\equiv T_{m,p}(x) \text{ in } (c, d), \\ &\equiv p(x) \text{ in } (a, c) \text{ and } (d, b). \end{aligned} \tag{42}$$

We can apply (33), which gives (since $(q-p)_{\max} = E_m(p)$)

$$|\varphi_n(p;x) - \varphi_n(q;x)| < \tau\sqrt{E_m(p)K_n(p;x)}, \tag{43}$$

where τ does not depend on x, nor on n, nor on m.*

We assume that m and n are increasing indefinitely, but $m/n \to 0$. Formula (43) combined with (40) (where $p(x)$, s, a, b must be replaced respectively by $q(x)$, m, c, d, and $\alpha = \beta = 1$, since $q(x) > 0$ for $c \leqq x \leqq d$) and with the results of Theorem I (page 540) gives, m and n being sufficiently large, the fundamental formula

$$\begin{gathered} |\varphi_n(p;z)| < \tau\left[\sqrt{m} + \sqrt{n E_m(p)}\right] \qquad (c+\varepsilon \leqq z \leqq d-\varepsilon), \\ |\varphi_n(p;c)|,\ |\varphi_n(p;d)| < \tau\sqrt{n}\left[\sqrt{m} + \sqrt{n E_m(p)}\right] \end{gathered} \tag{44}$$

under condition (41), *where τ does not depend on z, nor on n, nor on m, and $\varepsilon > 0$ is arbitrarily small, but fixed.*

In order to derive from (44) all the conclusions available, we take

$$m = \text{integral part of } n^{\beta} \text{ with } 0 < \beta < 1,\ n \to \infty.$$

* Formula (43) holds also for (a, b) infinitely large, provided (c, d) is finite.

Then

$$m \to \infty,\ n \to \infty,\ \frac{m}{n} \to 0,\ \frac{m}{n^{\beta}} \to 1,\ E_m(p) \to 0.$$

We can now use Table A (page 539), β being chosen so as to make the right-hand member in (44) of the highest order possible with respect to $1/n$. The results thus obtained are summarized in the following Table.

Table C: $\varphi_n(p; x)$

Conditions imposed on $p(x)$	z	$\varphi_n(p; z) =$
1. $p(x)$ is continuous and positive for $(a \leqq)\, c \leqq x \leqq d\, (\leqq b)$.	$c + \varepsilon \leqq z \leqq d - \varepsilon$	$o(n^{1/2})$
1, and 2. $\lvert p^{(k)}(x_2) - p^{(k)}(x_1) \rvert < \lambda \lvert x_2 - x_1 \rvert^{\alpha}$ for $c \leqq x_1,\ x_2 \leqq d$; $p^{(0)}(x) \equiv p(x)$	"	$O(n^{1/2(1+k+\alpha)})$*
(in particular for $k = 0$, $\alpha = 1$: Lipschitz condition).	"	$O(n^{1/4})$
1, and 3. $p^{(k)}(x)$ is continuous for $c \leqq x \leqq d$.	"	$O(n^{1/2(1+k)})$
1, and 4. $p(x)$ is indefinitely differentiable in (c, d).	"	$o(n^{\sigma})$, $\sigma > 0$ arbitrarily small
5–8. Same conditions as in 1–4 above.	$a = c \leqq z \leqq d = b$	1/2 must be added to each of the exponents of n in 1–4 above.

The Tables A, B, C, *as well as the results of* § 2 (*concerning* $K_n(p; x)$) *enable us to determine the convergence and the order* (*with respect to n*) *of the remainder of the development of a continuous function into a series according to Tchebycheff's polynomials of a given type.*

Many theorems can be formulated in this way. As an illustration, we state the following:

THEOREM III. *Suppose $f(x)$ is continous in a given finite interval (a, b) and satisfies the condition*

$$|f(x_2) - f(x_1)| \leqq \omega(\delta) \text{ for } |x_1 - x_2| \leqq \delta \qquad (a \leqq x_1,\ x_2 \leqq b).$$

* See G. Szegö, loc. cit., p. 139.

Then, in the development

$$f(x) = \sum_0^n A_i \varphi_i(x) + r_{n,f}(x), \qquad A_i = \int_a^b p(x) f(x) \varphi_i(x)\, dx,$$

where $\quad p(x) = (x-a)^{\alpha-1}(b-x)^{\beta-1}\Pi(x) \qquad (\alpha, \beta > 0),$

$\Pi(x)$ *a polynomial* $(\Pi(a)\Pi(b) \neq 0)$,

we have

$$r_{n,f}(z) = O\left(\omega\left(\frac{b-a}{n}\right)\log n\right)^*$$

at any point $x = z$ *inside* (a, b), *provided* $\Pi(z) \neq 0$. *In particular, the development under consideration converges to* $f(x)$ *uniformly for* $c + \varepsilon \leqq x \leqq d - \varepsilon$ $(a \leqq c;\ d \leqq b;\ \varepsilon > 0$ *arbitrarily small, but fixed), if* $f(x)$ *satisfies a Dini-Lipschitz condition, provided the interval* (c, d) *contains no roots of* $\Pi(x)$.

In the particular case $\alpha = \beta = 1$, $\Pi(x) \equiv 1$, we get the results obtained by D. Jackson, as was mentioned above.

---

* This follows from Table A_1, Table B_2 and p. 544.

The University of Chicago,
Chicago, Ill.

ANALYTIC FUNCTIONS IN THREE DIMENSIONS*

BY

E. R. HEDRICK AND LOUIS INGOLD

1. **Introduction.** In this paper the authors discuss certain analogies in three-dimensional space of the classical theory of analytic functions in two dimensions.

It is possible that isolated instances of such analogies have occured to many mathematicians, but so far as we are aware no systematic investigation of this subject has ever been published.

The subject is approached by means of the *stretching factor* of a transformation at a point and the related generalized Tissot indicatrix.

In two dimensions it has been shown that the Tissot indicatrix for an analytic function is a circle, and the condition that the Tissot indicatrix be a circle leads immediately to the Cauchy-Riemann equations.†

The corresponding condition in three dimensions‡ leads to equations analogous to the Cauchy-Riemann equations, and the corresponding function or transformation is conformal. Conformal transformations of space have, of course, received considerable attention, but their analogies with analytic functions in two dimensions do not appear to have been sufficiently emphasized.

A satisfactory generalization of analytic functions by way of the derivative property seems difficult. A definition of multiplication or the equivalent is needed and the commutative property is desirable.§

The derivative property may, however, be investigated from other angles, and some of these may possibly be extensible to three dimensions. In a

* Presented to the Society, September 7, 1923.

† See Hedrick, Ingold, and Westfall, *Theory of non-analytic functions of a complex variable*, Journal de Mathématiques, ser. 9, vol. 2 (1923), pp. 327-342.

‡ We shall discuss in this paper in detail only the case of equal axes. The general case is mentioned and will be discussed in detail in another paper. Some, but not all, of these details are similar to those developed in the paper by Hedrick, Ingold, and Westfall just cited.

§ The following definition of multiplication was considered: Given $W=(u,v,w)$, $R=(p,q,r)$, $WR=[u,v,w]\,[p,q,r]=[pu+qw+rv,\ pv+qu+rw,\ ru+pw+qv]$. According to this definition, the reciprocal of $Z=(x,y,z)$ is (a,b,c) where $a=(x^2-yz)/\Delta$, $b=(z^2-xy)/\Delta$, $c=(y^2-zx)/\Delta$, $\Delta=(x+y+z)\,(x^2+y^2+z^2-yz-zx-xy)$.

With this definition it is possible to define the derivative of $W=f(z)$ and the conditions for the (unique) existence of the derivative are easily obtained. The peculiar nature of ordinary singularities (poles) as revealed by the forms for a, b, c above seems to make a theory based on this definition undesirable. A similar objection has been noticed by G. Y. Rainich (Bulletin of the American Mathematical Society, vol. 30 (1924), p. 8) to a generalization based on quaternionic multiplication.

future paper the authors hope to give some results of investigations along this line; but for the present we devote our attention to the more fundamental analogies which spring from the generalization of the Cauchy-Riemann equations.

2. **The expansion factor in three dimensions.** As in two dimensions, a function

$$(u, v, w) = F(x, y, z) \tag{1}$$

corresponds to a transformation from the xyz-space to the uvw-space.

If arc length in the xyz-space is given by the equation

$$ds^2 = dx^2 + dy^2 + dz^2$$

and in the uvw space by

$$d\sigma^2 = du^2 + dv^2 + dw^2,$$

the expansion factor of the function $R = d\sigma^2/ds^2$ is easily computed. We have

$$du = u_x\, dx + u_y\, dy + u_z\, dz, \tag{2}$$

with similar expressions for dv and dw, and consequently

$$d\sigma^2 = E_{11}\, dx^2 + E_{22}\, dy^2 + E_{33}\, dz^2 + 2E_{12}\, dx\, dy + 2E_{23}\, dy\, dz + 2E_{31}\, dz\, dx, \tag{3}$$

where

$$E_{11} = u_x^2 + v_x^2 + w_x^2, \qquad E_{12} = u_x u_y + v_x v_y + w_x w_y, \text{ etc.}$$

Thus we have

$$R = \frac{E_{11}\, dx^2 + \cdots}{dx^2 + dy^2 + dz^2}. \tag{4}$$

3. **Stationary values of R.** The expansion factor R will, in general, have different values in different directions. The equations for the extremes of R, when they exist, may be written in the form

$$\begin{aligned} E_{11}\, dx + E_{12}\, dy + E_{13}\, dz &= R\, dx, \\ E_{21}\, dx + E_{22}\, dy + E_{23}\, dz &= R\, dy, \\ E_{31}\, dx + E_{32}\, dy + E_{33}\, dz &= R\, dz. \end{aligned} \tag{5}$$

Eliminating dx, dy, dz we obtain a cubic for R,

$$\begin{vmatrix} E_{11} - R & E_{12} & E_{13} \\ E_{21} & E_{22} - R & E_{23} \\ E_{31} & E_{32} & E_{33} - R \end{vmatrix} = 0. \tag{6}$$

Let R and ϱ be two distinct solutions of this cubic and let dx, dy, dz and $\delta x, \delta y, \delta z$ be the two sets of differentials corresponding to them. Multiply the equations (5) in order by $\delta x, \delta y, \delta z$ and add; then if we reduce by equations (5) applied to ϱ and $\delta x, \delta y, \delta z$, we find

$$dx(\varrho\,\delta x) + dy(\varrho\,\delta y) + dz(\varrho\,\delta z) = R(dx\,\delta x + dy\,\delta y + dz\,\delta z),$$

or

$$(\varrho - R)(dx\,\delta x + dy\,\delta y + dz\,\delta z) = 0, \tag{7}$$

and since ϱ and R are distinct this shows that the two corresponding directions are orthogonal. When the solutions of the cubic in R are all distinct the corresponding directions determined by equations (5) are also distinct and mutually orthogonal. We shall call them the *principal directions* of the function.

A curve whose direction at each point coincides with a principal direction is called a *characteristic curve* of the function. When the solutions of the cubic in R are distinct there are three mutually orthogonal families of characteristic curves.

4. Analytic functions. The case in which R has the same value in all directions corresponds to the analytic case in two dimensions. For this case it is necessary and sufficient that

$$\begin{aligned} R &= \frac{d\sigma^2}{ds^2} \\ &= \frac{E_{11}\,dx^2 + E_{22}\,dy^2 + E_{33}\,dz^2 + 2(E_{12}\,dx\,dy + E_{23}\,dy\,dz + E_{31}\,dz\,dx)}{dx^2 + dy^2 + dz^2} \end{aligned} \tag{8}$$

be independent of the direction, i. e., of dx, dy, dz. We can now determine the conditions on the E_{ij} by giving special values to dx, dy, dz or their ratios. If $dx = dy = 0$, and $dz = 1$, we have $E_{33} = R$. Similarly $E_{11} = R$, $E_{22} = R$. Using these values we have

$$R = \frac{E(dx^2 + dy^2 + dz^2) + 2(E_{12}\,dx\,dy + E_{23}\,dy\,dz + E_{31}\,dz\,dx)}{dx^2 + dy^2 + dz^2}$$

where E denotes the common value of E_{11}, E_{22} and E_{33}.

It follows that $E_{12}\,dx\,dy + E_{23}\,dy\,dz + E_{31}\,dz\,dx$ must vanish for all values of dx, dy, dz and hence

$$E_{12} = E_{23} = E_{31} = 0.$$

Thus the necessary and sufficient conditions* that R have the same value in all directions are that

$$(9) \qquad E_{11} = E_{22} = E_{33} = E, \text{ and } E_{12} = E_{23} = E_{31} = 0.$$

It is well known that the corresponding transformation is conformal.†

It is also known that the solutions of the equation $d\sigma^2 = E ds^2$ may always be obtained by a succession of inversions.‡

5. **Generalization of the Cauchy-Riemann equations.** From the above we have immediately (for the conformal case)

$$(10) \qquad \begin{aligned} u_x^2 + v_x^2 + w_x^2 &= E, \\ u_y^2 + v_y^2 + w_y^2 &= E, \\ u_z^2 + v_z^2 + w_z^2 &= E; \end{aligned}$$

$$(11) \qquad \begin{aligned} u_x u_y + v_x v_y + w_x w_y &= 0, \\ u_y u_z + v_y v_z + w_y w_z &= 0, \\ u_z u_x + v_z v_x + w_z w_x &= 0. \end{aligned}$$

From (11) we have

$$(12) \qquad u_x = \lambda \begin{vmatrix} v_y & w_y \\ v_z & w_z \end{vmatrix}, \qquad v_x = \lambda \begin{vmatrix} w_y & u_y \\ w_z & u_z \end{vmatrix}, \qquad w_x = \lambda \begin{vmatrix} u_y & v_y \\ u_z & v_z \end{vmatrix},$$

where λ is some multiplier, in general a function of x, y, z.

When these are substituted in the first of equations (10) it is found after reduction by means of the second and third of equations (10) that $\lambda = 1/\sqrt{E}$.

It is now easy to show that

$$(13) \qquad J = \begin{vmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{vmatrix} = u_x \begin{vmatrix} v_y & v_z \\ w_y & w_z \end{vmatrix} + u_y \begin{vmatrix} v_z & v_x \\ w_z & w_x \end{vmatrix} + u_z \begin{vmatrix} v_x & v_y \\ w_x & w_y \end{vmatrix} = E^{3/2},$$

---

* These conditions might be obtained by expressing the condition that the characteristic ellipsoid should reduce to a sphere.

† See Eisenhart, *Differential Geometry*, p. 99. The proof for three or more dimensions is wholly analogous to the proof for two dimensions.

‡ Sir William Thompson seems to have been the first to use the method of inversion to find functions u, v, w, satisfying the equation

$$du^2 + dv^2 + dw^2 = E(dx^2 + dy^2 + dz^2)$$

(see Liouville's Journal, vol. 10, p. 364). Later (Liouville's Journal, vol. 15, p. 103) Liouville announced that the only solutions were those given by Thompson's method. Professor Eisenhart has recently called our attention to Bianchi's generalization of Liouville's theorem. See *Lezioni di Geometria Differenziale*, 2d edition, vol. 1, pp. 375, 376.

and also that

$$
(14)\qquad \begin{aligned} u_x^2+u_y^2+u_z^2 &= v_x^2+v_y^2+v_z^2 = w_x^2+w_y^2+w_z^2 = E,\\ u_xv_x+u_yv_y+u_zv_z &= v_xw_x+v_yw_y+v_zw_z = w_xu_x+w_yu_y+w_zu_z = 0. \end{aligned}
$$

Equations (12) seem to be the natural generalizations of the Cauchy-Riemann equations. In complete form they are

$$
(15)\qquad \begin{aligned}
u_x &= \frac{1}{\sqrt{E}}\begin{vmatrix} v_y & w_y \\ v_z & w_z \end{vmatrix}, & v_x &= \frac{1}{\sqrt{E}}\begin{vmatrix} w_y & u_y \\ w_z & u_z \end{vmatrix}, & w_x &= \frac{1}{\sqrt{E}}\begin{vmatrix} u_y & v_y \\ u_z & v_z \end{vmatrix},\\
u_y &= \frac{1}{\sqrt{E}}\begin{vmatrix} v_z & w_z \\ v_x & w_x \end{vmatrix}, & v_y &= \frac{1}{\sqrt{E}}\begin{vmatrix} w_z & u_z \\ w_x & u_x \end{vmatrix}, & w_y &= \frac{1}{\sqrt{E}}\begin{vmatrix} u_z & v_z \\ u_x & v_x \end{vmatrix},\\
u_z &= \frac{1}{\sqrt{E}}\begin{vmatrix} v_x & w_x \\ v_y & w_y \end{vmatrix}, & v_z &= \frac{1}{\sqrt{E}}\begin{vmatrix} w_x & u_x \\ w_y & u_y \end{vmatrix}, & w_z &= \frac{1}{\sqrt{E}}\begin{vmatrix} u_x & v_x \\ u_y & v_y \end{vmatrix}.
\end{aligned}
$$

6. **Generalization of Laplace's equation.** From the generalized Cauchy-Riemann equations we have

$$
(16)\qquad \begin{vmatrix} v_y & w_y \\ v_z & w_z \end{vmatrix} = \sqrt{E}\,u_x, \quad \begin{vmatrix} v_z & w_z \\ v_x & w_x \end{vmatrix} = \sqrt{E}\,u_y, \quad \begin{vmatrix} v_x & w_x \\ v_y & w_y \end{vmatrix} = \sqrt{E}\,u_z,
$$

where

$$
E = u_x^2 + u_y^2 + u_z^2;
$$

and from these we obtain immediately (by taking derivatives and adding) the equation

$$
(17)\qquad \frac{\partial(\sqrt{E}\,u_x)}{\partial x} + \frac{\partial(\sqrt{E}\,u_y)}{\partial y} + \frac{\partial(\sqrt{E}\,u_z)}{\partial z} = 0.
$$

The functions v and w must satisfy the same differential equation if $(u, v, w) = F(x, y, z)$ represents a conformal transformation.

This equation may justly be regarded as a generalization of the equation $\partial^2 u/\partial x^2 + \partial^2 u/\partial y^2 = 0$ for the two-dimensional case.

The close analogy between this equation and the traditional Laplace equation both in two and in three dimensions is at once evident. This one reduces to the customary form if E is a constant.

In future papers we expect to continue this study of the analogies between functions in three dimensions and functions in two dimensions. In particular we expect to obtain extensions of the well known Beltrami equations, and to give still other generalizations of Laplace's equation with applications to the problem of conjugate functions.

THE UNIVERSITY OF CALIFORNIA, SOUTHERN BRANCH;
THE UNIVERSITY OF MISSOURI, COLUMBIA, MO.

THE BELTRAMI EQUATIONS IN THREE DIMENSIONS*

BY

E. R. HEDRICK AND LOUIS INGOLD

1. Introduction. The equations known as Beltrami's equations may be regarded as a generalization of the Cauchy-Riemann equations. The Cauchy-Riemann equations apply to functions of a complex variable in a plane while the Beltrami equations apply to functions of a complex variable in an arbitrary surface in three dimensions referred to any pair of curvilinear coördinates.

These equations may be obtained by transforming the Cauchy-Riemann equations to curvilinear coördinates: thus suppose (u, v) is a function (not necessarily analytic) of (x, y). Associated with this function are quantities E, F, G defined as in differential geometry:

$$E = u_x^2 + v_x^2, \qquad F = u_x u_y + v_x v_y, \qquad G = u_y^2 + v_y^2.$$

Thus the conditions that another function (U, V) of (x, y) be an analytic function of (u, v) are easily found to be

$$(1) \qquad V_x = \frac{1}{\sqrt{EG - F^2}} \begin{vmatrix} U_x & E \\ U_y & F \end{vmatrix}, \qquad V_y = \frac{1}{\sqrt{EG - F^2}} \begin{vmatrix} U_x & F \\ U_y & G \end{vmatrix}.$$

While these have the same form as Beltrami's equations the proof does not apply to surfaces in general, since E, F, and G are here the coefficients of a quadratic form $E\,dx^2 + 2F\,dx\,dy + G\,dy^2$ of curvature zero.

This method applied to functions in three dimensions leads to a generalization of Beltrami's equations, but here again the result would only be established for spaces of curvature zero.

The usual method for arbitrary two-dimensional surfaces makes use of the imaginary factorization of the quadratic form $E\,dx^2 + 2F\,dx\,dy + G\,dy^2$. Obviously this method does not admit of generalization to higher dimensions.

In order, then, to find the analogues of Beltrami's equations for arbitrary curved spaces of three dimensions (or higher) some new method must be devised although the form of these generalizations may possibly be suggested by the first of the methods mentioned above.

---

* Presented to the Society, Southwestern Section, December 1, 1923.

The case of space of zero curvature is of sufficient importance to deserve separate treatment. It is therefore first considered in this paper and by the method just indicated for the plane.

The case of curved spaces is then investigated and it is found that a different property of the transformations in question leads to equations of precisely the same form for this case. These equations are then used to obtain a classification of three-dimensional functions just as the Beltrami equations in the plane have been used to classify general functions of a complex variable.*

2. **Extension to ordinary space.** If the discussion is limited to ordinary space the generalization desired may be obtained for three dimensions, as well as for two, by combining an analytic and a non-analytic function. Let (u, v, w) be any function of (x, y, z) and (U, V, W) be an analytic function† of (u, v, w). Let us write

$$x = f(u, v, w),$$
$$y = \varphi(u, v, w),$$
$$z = \psi(u, v, w).$$

Regarding x, y, z as the independent variables and taking derivatives of both sides of each equation with respect to x, y, and z, we find

$$1 = x_u u_x + x_v v_x + x_w w_x, \quad 0 = x_u u_y + x_v v_y + x_w w_y, \quad 0 = x_u u_z + x_v v_z + x_w w_z,$$
$$0 = y_u u_x + y_v v_x + y_w w_x, \quad 1 = y_u u_y + y_v v_y + y_w w_y, \quad 0 = y_u u_z + y_v v_z + y_w w_z,$$
$$0 = z_u u_x + z_v v_x + z_w w_x, \quad 0 = z_u u_y + z_v v_y + z_w w_y, \quad 1 = z_u u_z + z_v v_z + z_w w_z.$$

Solving we obtain

$$u_x = J\begin{vmatrix} y_v & y_w \\ z_v & z_w \end{vmatrix}, \quad u_y = J\begin{vmatrix} z_v & z_w \\ x_v & x_w \end{vmatrix}, \quad u_z = J\begin{vmatrix} x_v & x_w \\ y_v & y_w \end{vmatrix},$$
$$v_x = J\begin{vmatrix} y_w & y_u \\ z_w & z_u \end{vmatrix}, \quad v_y = J\begin{vmatrix} z_w & z_u \\ x_w & x_u \end{vmatrix}, \quad v_z = J\begin{vmatrix} x_w & x_u \\ y_w & y_u \end{vmatrix},$$
$$w_x = J\begin{vmatrix} y_u & y_v \\ z_u & z_v \end{vmatrix}, \quad w_y = J\begin{vmatrix} z_u & z_v \\ x_u & x_v \end{vmatrix}, \quad w_z = J\begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix},$$

where J is the jacobian of (u, v, w) with respect to (x, y, z). Now because (U, V, W) is an analytic function of (u, v, w), we have

* See Hedrick, Ingold, and Westfall, *Theory of non-analytic functions of a complex variable*, Journal de Mathématiques, ser. 9, vol. 2 (1923).

† See a paper by the authors, pp. 551–555 of the present number of these Transactions.

$$U_u = \frac{1}{\sqrt{E}} \begin{vmatrix} V_x & V_w \\ W_v & W_w \end{vmatrix},$$

and so on; and remembering that u, v, w are functions of x, y, z we have

$$U_x x_u + U_y y_u + U_z z_u = \frac{1}{\sqrt{E}} \{[x_v\, y_w][V_x W_y] + [y_v\, z_w][V_y W_z] + [z_v\, x_w][V_z W_x]\},$$

$$U_x x_v + U_y y_v + U_z z_v = \frac{1}{\sqrt{E}} \{[x_w y_u][V_x W_y] + [y_w z_u][V_y W_z] + [z_w x_u][V_z W_x]\},$$

$$U_x x_w + U_y y_w + U_z z_w = \frac{1}{\sqrt{E}} \{[x_u y_v][V_x W_y] + [y_u z_v][V_y W_z] + [z_u x_v][V_z W_x]\},$$

where the small brackets on the right represent second order determinants, according to the usual notation.

These equations may be solved for U_x by multiplying in order by u_x, v_x, w_x and adding. The result on the right can be reduced by using the values for u_x, v_x, etc. found above. Similarly U_y and U_z can be found; and also

$$V_x,\ V_y,\ V_z, \text{ and } W_x,\ W_y,\ W_z.$$

The final results are given below; the E_{ij} are the coefficients of the differential form connecting (u, v, w) with (x, y, z) and E is the coefficient of the differential form connecting (U, V, W) with (u, v, w):*

(2)

$$U_x = \frac{1}{J\sqrt{E}} \begin{vmatrix} E_{11} & E_{12} & E_{13} \\ V_x & V_y & V_z \\ W_x & W_y & W_z \end{vmatrix}, \qquad U_y = \frac{1}{J\sqrt{E}} \begin{vmatrix} E_{21} & E_{22} & E_{23} \\ V_x & V_y & V_z \\ W_x & W_y & W_z \end{vmatrix},$$

$$U_z = \frac{1}{J\sqrt{E}} \begin{vmatrix} E_{31} & E_{32} & E_{33} \\ V_x & V_y & V_z \\ W_x & W_y & W_z \end{vmatrix},$$

$$V_x = \frac{1}{J\sqrt{E}} \begin{vmatrix} U_x & U_y & U_z \\ E_{11} & E_{12} & E_{13} \\ W_x & W_y & W_z \end{vmatrix}, \qquad V_y = \frac{1}{J\sqrt{E}} \begin{vmatrix} U_x & U_y & U_z \\ E_{21} & E_{22} & E_{23} \\ W_x & W_y & W_z \end{vmatrix},$$

$$V_z = \frac{1}{J\sqrt{E}} \begin{vmatrix} U_x & U_y & U_z \\ E_{31} & E_{32} & E_{33} \\ W_x & W_y & W_z \end{vmatrix},$$

* In this case only one of the fundamental quantities need be used, since, in the analytic case, $E_{11} = E_{22} = E_{33} = E$ and $E_{ij} = 0$ if $i \neq j$. See papers by the authors, loc. cit.

$$W_x = \frac{1}{J\sqrt{E}} \begin{vmatrix} U_x & U_y & U_z \\ V_x & V_y & V_z \\ E_{11} & E_{12} & E_{13} \end{vmatrix}, \qquad W_y = \frac{1}{J\sqrt{E}} \begin{vmatrix} U_x & U_y & U_z \\ V_x & V_y & V_z \\ E_{21} & E_{22} & E_{23} \end{vmatrix},$$

$$W_z = \frac{1}{J\sqrt{E}} \begin{vmatrix} U_x & U_y & U_z \\ V_x & V_y & V_z \\ E_{31} & E_{32} & E_{33} \end{vmatrix}.$$

These equations may also be written in the form

$$\text{(3)} \quad \begin{vmatrix} U_y & V_y \\ U_z & V_z \end{vmatrix} = \frac{\sqrt{E}}{J} \begin{vmatrix} W_x & E_{12} & E_{13} \\ W_y & E_{22} & E_{23} \\ W_z & E_{32} & E_{33} \end{vmatrix}, \qquad \begin{vmatrix} U_z & V_z \\ U_x & V_x \end{vmatrix} = \frac{\sqrt{E}}{J} \begin{vmatrix} E_{11} & W_x & E_{13} \\ E_{12} & W_y & E_{23} \\ E_{13} & W_z & E_{33} \end{vmatrix}, \text{ etc.}$$

3. **Properties of Beltrami's equations.** Writing out the values for U_x, V_x, W_x, we have*

$$\begin{aligned} U_x &= \frac{1}{J\sqrt{E}} \left\{ E_{11} \begin{vmatrix} V_y & V_z \\ W_y & W_z \end{vmatrix} + E_{12} \begin{vmatrix} V_z & V_x \\ W_z & W_x \end{vmatrix} + E_{13} \begin{vmatrix} V_x & V_y \\ W_x & W_y \end{vmatrix} \right\}, \\ \text{(4)} \qquad V_x &= \frac{1}{J\sqrt{E}} \left\{ E_{11} \begin{vmatrix} W_y & W_z \\ U_y & U_z \end{vmatrix} + E_{12} \begin{vmatrix} W_z & W_x \\ U_z & U_x \end{vmatrix} + E_{13} \begin{vmatrix} W_x & W_y \\ U_x & U_y \end{vmatrix} \right\}, \\ W_x &= \frac{1}{J\sqrt{E}} \left\{ E_{11} \begin{vmatrix} U_y & U_z \\ V_y & V_z \end{vmatrix} + E_{12} \begin{vmatrix} U_z & U_x \\ V_z & V_x \end{vmatrix} + E_{13} \begin{vmatrix} U_x & U_y \\ V_x & V_y \end{vmatrix} \right\}. \end{aligned}$$

Multiplying the first by U_x, the second by V_x, and the third by W_x and adding, we find

$$U_x^2 + V_x^2 + W_x^2 = \frac{J'}{J\sqrt{E}} E_{11},$$

where J' is the jacobian of (U, V, W) with respect to (x, y, z).

Similarly

$$U_x U_y + V_x V_y + W_x W_y = \frac{J'}{J\sqrt{E}} E_{12},$$

$$U_x U_z + V_x V_z + W_x W_z = \frac{J'}{J\sqrt{E}} E_{13}.$$

From the expanded forms for U_y, V_y, W_y, and U_z, V_z W_z, analogous formulas can be derived.

Thus if (U, V, W) is an analytic function of (u, v, w), the fundamental quantities of the function $(U, V, W) = \Phi(x, y, z)$ are proportional to the

* The special analytic case, in which $E_{11} = E_{22} = E_{33} = E$ and $E_{ij} = 0$ if $i \neq j$ obviously gives the Cauchy-Riemann equations of the earlier paper.

fundamental quantities of $(u, v, w) = F(x, y, z)$. This is a case of the extension of the Beltrami theorem that two "*functions on the same surface*" are analytic functions of each other. The general case is given in § 5. It is clear also that two different analytic functions of (u, v, w) have fundamental quantities that are proportional when both are regarded as functions of (x, y, z).

4. **Extension to curved spaces.** Consider a curved surface with the fundamental quantities E, F, G; let $\boldsymbol{t}_1$ and $\boldsymbol{t}_2$ be tangent vectors to the parameter curves such that $\boldsymbol{t}_1 \boldsymbol{t}_2 = F$, $\boldsymbol{t}_1 \boldsymbol{t}_1 = E$, $\boldsymbol{t}_2 \boldsymbol{t}_2 = G$.

If $u(x, y) = c$ determines a family of curves on the surface, the vector

$$\frac{\partial u}{\partial x}\boldsymbol{t}_2 - \frac{\partial u}{\partial y}\boldsymbol{t}_1$$

is at each point tangent to $u = c$. If $v = k$ is another family of curves on the surface, the vector

$$\left(F\frac{\partial v}{\partial y} - G\frac{\partial v}{\partial x}\right)\boldsymbol{t}_1 + \left(F\frac{\partial v}{\partial x} - E\frac{\partial v}{\partial y}\right)\boldsymbol{t}_2$$

is at each point orthogonal to $v = k$. The condition that $u = c$ be the orthogonal trajectories of $v = k$ is

$$F\frac{\partial v}{\partial y} - G\frac{\partial v}{\partial x} = -h\frac{\partial u}{\partial y}, \qquad F\frac{\partial v}{\partial x} - E\frac{\partial v}{\partial y} = h\frac{\partial u}{\partial x},$$

where h is a scalar factor.

If, in addition, the two invariants* $\Delta_1 u$, $\Delta_1 v$ are to be equal it is found that h must be equal to $\sqrt{EG - F^2}$. Thus the necessary and sufficient conditions that $u = c$ and $v = k$ are the orthogonal trajectories of each other and that $\Delta_1 u = \Delta_1 v$ are

$$\frac{\partial u}{\partial y} = \frac{1}{\sqrt{EG - F^2}}\begin{vmatrix} F & \dfrac{\partial v}{\partial x} \\ G & \dfrac{\partial v}{\partial y}\end{vmatrix}, \qquad \frac{\partial u}{\partial x} = \frac{1}{\sqrt{EG - F^2}}\begin{vmatrix} E & \dfrac{\partial v}{\partial x} \\ F & \dfrac{\partial v}{\partial y}\end{vmatrix},$$

and these are Beltrami's equations. We now propose a similar problem in three dimensions: Consider a curved, three-dimensional space with the

* $\Delta_1 \Phi$ is the differential parameter

$$\left[E\left(\frac{\partial \Phi}{\partial y}\right)^2 - 2F\frac{\partial \Phi}{\partial x}\,\frac{\partial \Phi}{\partial y} + G\left(\frac{\partial \Phi}{\partial x}\right)^2\right] \div (EG - F).$$

fundamental quantities E_{ij}; also let t_1, t_2, t_3 be three vectors tangent to the parameter curves such that $t_i t_j = E_{ij}$. The tangent vector to the curve of intersection of two surfaces $U = c$, $V = k$ can be written in the form

$$\begin{vmatrix} U_y & V_y \\ U_z & V_z \end{vmatrix} t_1 + \begin{vmatrix} U_z & V_z \\ U_x & V_x \end{vmatrix} t_2 + \begin{vmatrix} U_x & V_x \\ U_y & V_y \end{vmatrix} t_3 .$$

The normal vector to a surface $W = h$ has the form

$$\begin{vmatrix} W_x & E_{12} & E_{13} \\ W_y & E_{22} & E_{23} \\ W_z & E_{32} & E_{33} \end{vmatrix} t_1 + \begin{vmatrix} E_{11} & W_x & E_{13} \\ E_{12} & W_y & E_{23} \\ E_{13} & W_z & E_{33} \end{vmatrix} t_2 + \begin{vmatrix} E_{11} & E_{21} & W_x \\ E_{12} & E_{22} & W_y \\ E_{13} & E_{23} & W_z \end{vmatrix} t_3 .$$

Hence the necessary and sufficient condition that the surfaces $W = h$ be the orthogonal trajectories of the curves $U = c$, $V = k$ are given by the equations

$$\begin{vmatrix} W_x & E_{12} & E_{13} \\ W_y & E_{22} & E_{23} \\ W_z & E_{32} & E_{33} \end{vmatrix} = p \begin{vmatrix} U_y & V_y \\ U_z & V_z \end{vmatrix}, \quad \begin{vmatrix} E_{11} & W_x & E_{13} \\ E_{12} & W_y & E_{23} \\ E_{13} & W_z & E_{33} \end{vmatrix} = p \begin{vmatrix} U_z & V_z \\ U_x & V_x \end{vmatrix}, \quad \begin{vmatrix} E_{11} & E_{21} & W_x \\ E_{12} & E_{22} & W_y \\ E_{13} & E_{23} & W_z \end{vmatrix} = p \begin{vmatrix} U_x & V_x \\ U_y & V_y \end{vmatrix},$$

where p is a factor of proportionality.

These three equations may be solved for W_x, W_y, W_z. The resulting values are

$$W_x = P \begin{vmatrix} U_x & U_y & U_z \\ V_x & V_y & V_z \\ E_{11} & E_{12} & E_{13} \end{vmatrix}, \qquad W_y = P \begin{vmatrix} U_x & U_y & U_z \\ V_x & V_y & V_z \\ E_{21} & E_{22} & E_{23} \end{vmatrix}, \qquad W_z = P \begin{vmatrix} U_x & U_y & U_z \\ V_x & V_y & V_z \\ E_{31} & E_{32} & E_{33} \end{vmatrix},$$

where P is a new factor of proportionality.

If it is required that $U = c$ be the orthogonal trajectories of the curves $V = k$, $W = h$, and that $V = k$ be the orthogonal trajectories of $W = h$, $U = c$, we obtain

$$U_x = P \begin{vmatrix} E_{11} & E_{12} & E_{13} \\ V_x & V_y & V_z \\ W_x & W_y & W_z \end{vmatrix}, \qquad U_y = P \begin{vmatrix} E_{21} & E_{22} & E_{23} \\ V_x & V_y & V_z \\ W_x & W_y & W_z \end{vmatrix}, \qquad U_z = P \begin{vmatrix} E_{31} & E_{32} & E_{33} \\ V_x & V_y & V_z \\ W_x & W_y & W_z \end{vmatrix},$$

$$V_x = P \begin{vmatrix} U_x & U_y & U_z \\ E_{11} & E_{12} & E_{13} \\ W_x & W_y & W_z \end{vmatrix}, \qquad W_y = P \begin{vmatrix} U_x & U_y & U_z \\ E_{21} & E_{22} & E_{23} \\ W_x & W_y & W_z \end{vmatrix}, \qquad W_z = P \begin{vmatrix} U_x & U_y & U_z \\ E_{31} & E_{32} & E_{33} \\ W_x & W_y & W_z \end{vmatrix}.$$

These are the Beltrami equations for a curved space.

5. **Functions in curved space.** It is easy to show as in § 3 that

$$\sum U_x^2 = QE_{11}, \qquad \sum U_y^2 = QE_{22}, \qquad \sum U_z^2 = QE_{33},$$
$$\sum U_x U_y = QE_{12}, \qquad \sum U_y U_z = QE_{23}, \qquad \sum U_z U_x = QE_{31},$$

provided that U, V, and W satisfy Beltrami's equations, where Q is a factor of proportionality whose value can be found readily. Similar results hold for a function U', V', W' except that the proportionality factor Q' may be different from Q; consequently if (U, V, W) and (U', V', W') are two functions satisfying Beltrami's equations, we have

$$\sum U_x^2 = R\sum U_x'^2, \qquad \sum U_x U_y = R\sum U_x' U_y',$$
$$\sum U_y^2 = R\sum U_y'^2, \qquad \sum U_y U_z = R\sum U_y' U_z',$$
$$\sum U_z^2 = R\sum U_z'^2, \qquad \sum U_z U_x = R\sum U_z' U_x'.$$

Thus (U, V, W) is an analytic function of (U', V', W'), by § 4 of the preceding paper (p. 553). These equations constitute the essential generalization of the Beltrami theorem mentioned in § 3.

6. **Non-analytic functions.** Let u, v, w be any function $F(x, y, z)$ in ordinary space, and denote the fundamental quantities of this function by E_{ij}. By § 3 the fundamental quantities of $(U, V, W) = \Phi(x, y, z)$ are proportional to E_{ij} provided (U, V, W) is an analytic function of (u, v, w).

The converse of this is also true. If the fundamental quantities of $\Phi(x, y, z)$ are proportional to the fundamental quantities of the function $F(x, y, z)$, then Φ is an analytic function of F. Thus the ratios of the fundamental quantities E_{ij} determine a class of functions which are analytic functions of each other.*

With suitable conditions as to continuity and differentiability the following statements are easily proved:

Every function belongs to a definite class in a given region.

No function belongs to two different classes in the same region.

The totality of analytic functions in a given region constitutes a separate class.

There exist functions belonging to every class in a given region.

---

* This is the extension to three dimensions of the method of classification of functions of a complex variable given in the paper *Non-analytic functions*, etc., loc. cit.

The University of California, Southern Branch; | The University of Missouri, Columbia, Mo.

FIELDS OF PARALLEL VECTORS IN A RIEMANNIAN GEOMETRY*

BY

LUTHER PFAHLER EISENHART

1. **Introduction.** Consider a space V_n of n dimensions whose fundamental quadratic form is

$$\varphi = g_{ij}\, dx^i\, dx^j \qquad (i, j = 1, \cdots, n), \tag{1.1}$$

where the g's are subject only to the condition that their determinant

$$g = |g_{ij}| \tag{1.2}$$

is not zero.† We do not impose the condition that φ is definite. We define the metric by

$$ds^2 = e g_{ij}\, dx^i\, dx^j, \tag{1.3}$$

where e is plus or minus one according as φ is positive or negative for a given set of values of the differentials.

Consider any curve C in V_n which is not of length zero, that is for which φ is not zero, and assume that the coördinates x^i at points of C are functions of s measured from some point of C along it. Let λ^i be the contravariant components of a unit vector field in V_n, that is

$$g_{ij}\, \lambda^i \lambda^j = e,$$

where e is plus or minus one. In accordance with the concept of parallelism in V_n due to Levi-Civita, we say that the vectors of the field at points of C are parallel to one another with respect to C, if

$$\frac{dx^k}{ds}\lambda^i_{,k} \equiv \frac{dx^k}{ds}\left(\frac{\partial \lambda^i}{\partial x^k} + \left\{ {i \atop j\,k} \right\} \lambda^j\right) = 0, \tag{1.4}$$

where $\lambda^i_{,k}$ is the covariant derivative of λ^i, $\left\{ {i \atop j\,k} \right\}$ being the Christoffel symbols of the second kind formed with respect to (1.1). When

* Presented to the Society, February 28, 1925.

† We use the customary convention of summation indicated by a repeated index.

$$\lambda^i_{\ ,k} = 0, \tag{1.5}$$

and only in this case, any two vectors are parallel to one another with respect to any curve passing through their points. In this case we have by definition a *field of parallel vectors.*

Fields of parallel vectors for geometries of paths have been considered by the author* and by Veblen and Thomas;† the results of the latter can be used to give in invariantive form the conditions that a Riemannian space shall admit one or more fields of parallel vectors. In the present paper we derive the canonical forms of (1.1) for such spaces.‡

2. **Canonical coördinate system.** Suppose that a V_n is such that equations (1.5) admit p linearly independent sets of solutions, $\lambda_{\sigma|}{}^i$ for $\sigma = 1, \cdots, p$ and $i = 1, \cdots, n$.§ In any other coördinate system, x'^i, the components $\lambda'_{\sigma|}{}^i$ are given by

$$\lambda'_{\sigma|}{}^i = \lambda_{\sigma|}{}^j \frac{\partial x'^i}{\partial x^j}. \tag{2.1}$$

If we take the system of p differential equations

$$X_\sigma(\theta) \equiv \lambda_{\sigma|}{}^j \frac{\partial \theta}{\partial x^j} = 0 \tag{2.2}$$

and apply the Poisson operator, we have in consequence of (1.5)

$$(X_\sigma X_\tau - X_\tau X_\sigma)\,\theta \equiv 0.$$

Therefore equations (2.2) form a complete system, and consequently admit $n-p$ independent solutions.|| If we take them for the coördinates $x'^{p+1}, \cdots, x'^n$, it follows from (2.1) that $\lambda'_{\sigma|}{}^t = 0$ for $t = p+1, \cdots, n$. Again if we omit one of the equations from (2.2), say $X_r\theta = 0$, the remaining system is complete and admits in addition to $x'^{p+1}, \cdots, x'^n$ another independent solution x'^r. In this way the x''s are defined so that all of the components of the λ's are zero except those with the same

* Proceedings of the National Academy of Sciences, vol. 8 (1922), pp. 207–212.

† These Transactions, vol. 25 (1923), p. 589.

‡ The case of spaces admitting one field of parallel vectors was considered by Levi-Civita, Rendiconti del Circolo Matematico di Palermo, vol. 42 (1917), pp. 173–205.

§ Unless otherwise stated, it will be assumed that Greek indices take the values $1, \cdots, p$ and Latin $1, \cdots, n$, throughout this paper.

|| Goursat, *Leçons sur l'Intégration des Equations aux Dérivées Partielles du Premier Ordre*, p. 52.

subscript and superscript. If it is assumed that these vectors are unit vectors, we have in the new coördinate system

$$(2.3)\qquad \lambda_{\sigma|}{}^{\sigma} = \frac{1}{\sqrt{e_\sigma g_{\sigma\sigma}}}, \quad \lambda_{\sigma|}{}^{t} = 0 \quad (\sigma = 1, \cdots, p;\ t = 1, \cdots, n;\ t \neq \sigma).$$

When these expressions are substituted in (1.5) we get

$$\frac{\partial}{\partial x^k} \log \sqrt{g_{\sigma\sigma}} - \left\{ {\sigma \atop k\ \sigma} \right\} = 0 \qquad (\sigma \text{ not summed}),$$

$$\left\{ {j \atop k\ \sigma} \right\} = 0 \quad (\sigma = 1, \cdots, p;\ j, k = 1, \cdots, n;\ j \neq \sigma).$$

If we multiply the first of these equations by $g_{\sigma l}$ and subtract the second multiplied by g_{jl} and summed for j, we get the equivalent set of equations

$$g_{\sigma l} \frac{\partial}{\partial x^k} \log \sqrt{g_{\sigma\sigma}} - [k\sigma, l] = 0,$$

that is

$$(2.4)\qquad g_{\sigma l} \frac{\partial}{\partial x^k} \log g_{\sigma\sigma} - \frac{\partial g_{kl}}{\partial x^\sigma} - \frac{\partial g_{\sigma l}}{\partial x^k} + \frac{\partial g_{k\sigma}}{\partial x^l} = 0.$$

For the case $k = \sigma$, these equations reduce to

$$\frac{\partial}{\partial x^\sigma} \left(\frac{g_{\sigma l}}{\sqrt{e_\sigma g_{\sigma\sigma}}} \right) = e_\sigma \frac{\partial}{\partial x^l} \sqrt{e_\sigma g_{\sigma\sigma}}.$$

In accordance with these equations we define p functions ψ_σ by

$$\sqrt{e_\sigma g_{\sigma\sigma}} = e_\sigma \frac{\partial \psi_\sigma}{\partial x^\sigma}, \qquad \frac{g_{\sigma l}}{\sqrt{e_\sigma g_{\sigma\sigma}}} = \frac{\partial \psi_\sigma}{\partial x^l},$$

from which we have

$$(2.5)\qquad g_{\sigma l} = e_\sigma \frac{\partial \psi_\sigma}{\partial x^\sigma} \frac{\partial \psi_\sigma}{\partial x^l}.$$

From these expressions it follows that ψ_σ must involve x^σ, otherwise the space is of less than n dimensions.

Again if neither k nor l in (2.4) is σ, we have

$$(2.6)\qquad g_{kl} = e_\sigma \frac{\partial \psi_\sigma}{\partial x^k} \frac{\partial \psi_\sigma}{\partial x^l} + \varphi_{kl\sigma} \quad \left({\sigma = 1, \cdots, p;\ k, l = 1, \cdots, n; \atop k \neq \sigma,\ l \neq \sigma} \right),$$

where $\varphi_{kl\sigma}$ is a function independent of x^σ.

From (2.5) and (2.6) it follows that for each value of σ the fundamental form can be written

$$\varphi = e_\sigma (d\psi_\sigma)^2 + g_{rs}\, dx^r\, dx^s \quad (r, s = 1, \cdots, n;\ r \neq \sigma,\ s \neq \sigma),$$

where g_{rs} are independent of x^σ. If then we put $x'^\sigma = \psi^\sigma$, $x'^j = x^j\ (j \neq \sigma)$, the curves of parameter x'^σ are the same as those of parameter x^σ, and these curves are geodesics.* Hence we have:

When a V_n admits p independent fields of parallel vectors, the vectors of each field are the tangent vectors to a congruence of geodesics.

Conversely if the fundamental form of any space is reducible to the form

$$\varphi = e_1\, dx_1^2 + g_{rs}\, dx^r\, dx^s \qquad (r, s = 2, \cdots, n), \tag{2.7}$$

it is found from (2.4) that a necessary and sufficient condition that the tangents to the curves of parameter x^1 form a parallel field is that g_{rs} be independent of x^1. In this case all the spaces $x^1 = \text{const.}$ have the same fundamental form and consequently any one of them can be brought into coincidence with any other by a *translation*, that is by a motion in which each point describes the same distance along the geodesic normal to the sub-space. In the case $p > 1$ the space admits p independent translations; thus any two of the sub-spaces of each of the family of sub-spaces $\psi_\sigma = \text{const.}$ can be brought into coincidence by a translation.

If in particular we take $\psi_\sigma = x^\sigma + \varphi_\sigma(x^{p+1}, \cdots, x^n)$ for $\sigma = 1, \cdots, p$, it follows from (2.5) and (2.6) that for a V_n with the fundamental form

$$\varphi = e_1 (dx^1)^2 + \cdots + e_p (dx^p)^2 + g_{\alpha\beta}\, dx^\alpha\, dx^\beta \quad (\alpha, \beta = p+1, \cdots, n), \tag{2.8}$$

where $g_{\alpha\beta}$ are arbitrary functions of $x^{p+1}, \cdots, x^n$, the tangents to curves of parameters $x^1, \cdots, x^p$ form p fields of parallel vectors. We proceed to the determination of the general solution.

3. **When $p > 1$.** When $p > 1$, we have from (2.5), on equating the two expressions for $g_{\sigma\tau}$,

$$e_\sigma \frac{\partial \psi_\sigma}{\partial x^\sigma} \frac{\partial \psi_\sigma}{\partial x^\tau} = e_\tau \frac{\partial \psi_\tau}{\partial x^\sigma} \frac{\partial \psi_\tau}{\partial x^\tau}, \tag{3.1}$$

and from (2.5) and (2.6), on equating the two expressions for $g_{\tau\tau}$ and $g_{\sigma\sigma}$ respectively,

$$\begin{aligned} e_\sigma \left(\frac{\partial \psi_\sigma}{\partial x^\tau}\right)^2 + \varphi_{\tau\tau\sigma} &= e_\tau \left(\frac{\partial \psi_\tau}{\partial x^\tau}\right)^2, \\ e_\tau \left(\frac{\partial \psi_\tau}{\partial x^\sigma}\right)^2 + \varphi_{\sigma\sigma\tau} &= e_\sigma \left(\frac{\partial \psi_\sigma}{\partial x^\sigma}\right)^2. \end{aligned} \tag{3.2}$$

---

* Bianchi, *Lezioni*, vol. 1, p. 337.

When these equations are differentiated with respect to x^σ and x^τ respectively, we obtain

$$e_\sigma \frac{\partial \psi_\sigma}{\partial x^\tau} \frac{\partial^2 \psi_\sigma}{\partial x^\sigma \partial x^\tau} - e_\tau \frac{\partial \psi_\tau}{\partial x^\tau} \frac{\partial^2 \psi_\tau}{\partial x^\sigma \partial x^\tau} = 0,$$

$$e_\sigma \frac{\partial \psi_\sigma}{\partial x^\sigma} \frac{\partial^2 \psi_\sigma}{\partial x^\sigma \partial x^\tau} - e_\tau \frac{\partial \psi_\tau}{\partial x^\sigma} \frac{\partial^2 \psi_\tau}{\partial x^\sigma \partial x^\tau} = 0.$$

Hence either

$$\frac{\partial^2 \psi_\sigma}{\partial x^\sigma \partial x^\tau} = \frac{\partial^2 \psi_\tau}{\partial x^\sigma \partial x^\tau} = 0, \tag{3.3}$$

or

$$\frac{\partial \psi_\sigma}{\partial x^\tau} \frac{\partial \psi_\tau}{\partial x^\sigma} = \frac{\partial \psi_\sigma}{\partial x^\sigma} \frac{\partial \psi_\tau}{\partial x^\tau}. \tag{3.4}$$

If (3.4) holds, it follows from (3.1) and (3.2) that $\varphi_{\sigma\sigma\tau} = \varphi_{\tau\tau\sigma} = 0$. This case will be considered in § 7.

From equations of the form (3.3) it follows that the functions ψ_σ must be of the form

$$\begin{aligned} \psi_\sigma = {} & e_\sigma f_\sigma (x^\sigma, x^{p+1}, \cdots, x^n) \\ & + F_\sigma (x^1, \cdots, x^{\sigma-1}, x^{\sigma+1}, \cdots, x^p, x^{p+1}, \cdots, x^n) \quad (\sigma = 1, \cdots, p). \end{aligned} \tag{3.5}$$

From the remark following (2.5) it is seen that f_σ must involve x^σ. Moreover it is understood that f_σ does not have any additional function independent of x^σ, since all such terms are included in F_σ.

If $\partial \psi_\sigma / \partial x^\tau = 0$, so also is $\partial \psi_\tau / \partial x^\sigma = 0$, as follows from (3.1). These exceptional cases are taken care of by (3.5).

Substituting from (3.5) in (3.1), we get

$$\frac{\dfrac{\partial F_\tau}{\partial x^\sigma}}{\dfrac{\partial f_\sigma}{\partial x^\sigma}} = \frac{\dfrac{\partial F_\sigma}{\partial x^\tau}}{\dfrac{\partial f_\tau}{\partial x^\tau}}. \tag{3.6}$$

Since the first and second members of this equation are independent of x^τ and x^σ respectively, they are independent of both. Consequently we replace (3.6) by

$$\frac{\partial F_\tau}{\partial x^\sigma} = \frac{\partial f_\sigma}{\partial x^\sigma} \omega_{\tau\sigma}, \quad \frac{\partial F_\sigma}{\partial x^\tau} = \frac{\partial f_\tau}{\partial x^\tau} \omega_{\sigma\tau} \quad (\sigma, \tau = 1, \cdots, p;\ \sigma \neq \tau), \tag{3.7}$$

where $\omega_{\tau\sigma} (= \omega_{\sigma\tau})$ is independent of x^σ and x^τ.

4. **When $p=2$.** When $p=2$, we have from (3.5) and (3.7), since $\omega_{\sigma\tau}$ is symmetric,

$$\psi_\sigma = e_\sigma f_\sigma + A f_\tau + B_\sigma \qquad (\sigma, \tau = 1, 2; \sigma \neq \tau), \tag{4.1}$$

where A, B_1 and B_2 are independent of x^1 and x^2.

For the general case we have from (2.5) and the expressions for $g_{\sigma l}$ from (2.6) the equations of condition

$$e_\sigma \frac{\partial \psi_\sigma}{\partial x^\sigma} \frac{\partial \psi_\sigma}{\partial x^l} = e_\tau \frac{\partial \psi_\tau}{\partial x^\sigma} \frac{\partial \psi_\tau}{\partial x^l} + \varphi_{\sigma l \tau} \quad \begin{pmatrix} \sigma = 1, \cdots, p; l = 1, \cdots, n; \\ l, \sigma, \tau \neq \end{pmatrix}. \tag{4.2}$$

When $p=2$ and we take $\sigma=1$, $\tau=2$, $l=3, \cdots, n$, this condition for the expressions (4.1) becomes

$$\frac{\partial f_1}{\partial x^1}\left[f_2 \frac{\partial A}{\partial x^l} + \frac{\partial}{\partial x^l}(e_1 f_1 + B_1) - e_2 A_2 \frac{\partial}{\partial x^l}(A f_1 + B_2)\right] - \varphi_{1\sigma 2} = 0.$$

Since $\partial f_1/\partial x^1 \neq 0$ and all the terms except f_2 are independent of x^2, it follows that A must be a constant.

There remain for consideration the equations which arise from (2.6) on equating the expressions for each g_{kl} $(k, l = 3, \cdots, n)$ for $\sigma = 1, 2$. Since φ_{kl1} and φ_{kl2} may be any functions not involving x^1 and x^2 respectively, we find that these conditions are satisfied, because A is a constant. In arriving at this result we take

$$\begin{aligned} \varphi_{kl1} = {} & e_2 \frac{\partial}{\partial x^k}(e_2 f_2 + B_2)\frac{\partial}{\partial x^l}(e_2 f_2 + B_2) - e_1 \frac{\partial}{\partial x^k}(A f_2 + B_1)\frac{\partial}{\partial x^l}(A f_2 + B_1) \\ & + e_1 \frac{\partial B_1}{\partial x^k}\frac{\partial B_1}{\partial x^l} + C_{kl}, \end{aligned} \tag{4.3}$$

where C_{kl} are arbitrary functions of $x^3, \cdots, x^n$. Then we have

$$\begin{aligned} g_{kl} = {} & \sum_\sigma^{1,2} e_\sigma \frac{\partial}{\partial x^k}(e_\sigma f_\sigma + B_\sigma)\frac{\partial}{\partial x^l}(e_\sigma f_\sigma + B_\sigma) \\ & + A\left(\frac{\partial f_1}{\partial x^k}\frac{\partial f_2}{\partial x^l} + \frac{\partial f_1}{\partial x^l}\frac{\partial f_2}{\partial x^k}\right) + C_{kl}. \end{aligned} \tag{4.4}$$

Hence we have the following theorem:

When f_1 and f_2 are arbitrary functions of $x^1, x^3, \cdots, x^n$ and $x^2, x^3, \cdots, x^n$ respectively, A is an arbitrary constant, and B_1 and B_2 are arbitrary func-

tions of $x^3, \cdots, x^n$, *the functions* (4.1) *determine a* V_n *with two fields of parallel vectors; and* C_{kl} *in* (4.4) *are also arbitrary functions of* $x^3, \cdots, x^n$.

5. **When** $p = 3$. If $p > 2$ and we express the condition of integrability of two of the first set of equations (3.7) for two values of σ different from τ, say α and β, we find by considerations similar to those applied to (3.6)

$$\frac{\partial \omega_{\tau\alpha}}{\partial x^\beta} = \frac{\partial f_\beta}{\partial x^\beta}\omega_{\tau\alpha\beta}, \quad \frac{\partial \omega_{\tau\beta}}{\partial x^\alpha} = \frac{\partial f_\alpha}{\partial x^\alpha}\omega_{\tau\beta\alpha} \qquad \begin{pmatrix} \alpha, \beta, \tau = 1, \cdots, p; \\ \alpha, \beta, \tau \neq \end{pmatrix},$$

where $\omega_{\tau\alpha\beta} (= \omega_{\tau\beta\alpha})$ is independent of x^τ, x^α and x^β, and consequently

$$\omega_{\tau\alpha} = f_\beta\, \omega_{\tau\alpha\beta} + \varrho_{\tau\alpha\beta}, \tag{5.1}$$

where $\varrho_{\tau\alpha\beta}$ is independent of x^τ, x^α and x^β. Since $\omega_{\tau\alpha} = \omega_{\alpha\tau}$, $\omega_{\tau\alpha\beta}$ must be symmetric in τ, α and β, and $\varrho_{\tau\alpha\beta}$ in τ and α. For $p = 3$, we have from (3.7) and (5.1)

$$F_\sigma = A \prod_\alpha{}'_\sigma (f_\alpha) + \sum_\alpha{}'_\sigma B_{\sigma\alpha} f_\alpha + C_\sigma \qquad (\sigma = 1, 2, 3), \tag{5.2}$$

where $\prod_\alpha{}'_\sigma (f_\alpha)$ means the product of all the f's except f_σ and $\sum_\alpha{}'_\sigma$ indicates the sum for α except $\alpha = \sigma$; also A, $B_{\sigma\alpha} = B_{\alpha\sigma}$ and C_σ are functions independent of x^1, x^2 and x^3.

If we substitute in (4.2) the expressions $\psi_\sigma = e_\sigma f_\sigma + F_\sigma$, where F_σ is given by (5.2) and similarly for ψ_τ, we obtain equations of the form

$$\frac{\partial f_\sigma}{\partial x^\sigma} f_\tau \frac{\partial}{\partial x^l}(f_\varrho A + B_{\sigma\tau}) + \Phi_\tau = 0 \quad \begin{pmatrix} \varrho, \sigma, \tau = 1, 2, 3; & \varrho, \sigma, \tau \neq \\ l = 1, \cdots, n; & l, \sigma, \tau \neq \end{pmatrix},$$

where Φ_τ does not involve x^τ. Hence

$$\frac{\partial}{\partial x^l}(f_\varrho A + B_{\sigma\tau}) = 0.$$

For $l = \varrho$ we get $A = 0$, and for other values of l it follows that the B's are constant.

When as in § 4 we equate the three values of each g_{kl} $(k, l = 4, \cdots, n)$ obtained from (2.6) by taking $\sigma = 1, 2, 3$, we find that these equations are consistent in view of the above results. Moreover, the functions $\varphi_{kl\sigma}$ are of the forms

$$
\begin{aligned}
\varphi_{kl\sigma} &= e_\tau \frac{\partial}{\partial x^k}(e_\tau f_\tau + B_{\tau\varrho} f_\varrho + C_\tau) \frac{\partial}{\partial x^l}(e_\tau f_\tau + B_{\tau\varrho} f_\varrho + C_\tau) \\
&\qquad - e_\alpha B^2_{\sigma\tau} \frac{\partial f_\tau}{\partial x^k} \frac{\partial f_\tau}{\partial x^l} + D_{kl\sigma\tau} \\
&= e_\varrho \frac{\partial}{\partial x^k}(e_\varrho f_\varrho + B_{\tau\varrho} f_\tau + C_\varrho) \frac{\partial}{\partial x^l}(e_\varrho f_\varrho + B_{\tau\varrho} f_\tau + C_\varrho) \\
&\qquad - e_\alpha B^2_{\sigma\varrho} \frac{\partial f_\varrho}{\partial x^k} \frac{\partial f_\varrho}{\partial x^l} + D_{kl\sigma\varrho} \\
&\qquad (\varrho, \sigma, \tau = 1, 2, 3; \quad \varrho, \sigma, \tau \neq),
\end{aligned} \tag{5.3}
$$

where $D_{kl\sigma\tau}\ (= D_{kl\tau\sigma})$ is independent of x^σ and x^τ. These two expressions for $\varphi_{kl\sigma}$ are seen to be consistent, if we take

$$
\begin{aligned}
D_{kl\sigma\tau} &= e_\varrho \frac{\partial f_\varrho}{\partial x^k} \frac{\partial f_\varrho}{\partial x^l} + \frac{\partial f_\varrho}{\partial x^k} \frac{\partial C_\varrho}{\partial x^l} + \frac{\partial f_\varrho}{\partial x^l} \frac{\partial C_\varrho}{\partial x^k} \\
&\qquad - e_\tau \frac{\partial}{\partial x^k}(B_{\tau\varrho} f_\varrho + C_\tau) \frac{\partial}{\partial x^l}(B_{\tau\varrho} f_\varrho + C_\tau) \\
&\qquad - e_\sigma \frac{\partial}{\partial x^k}(B_{\sigma\varrho} f_\varrho + C_\sigma) \frac{\partial}{\partial x^l}(B_{\sigma\varrho} f_\varrho + C_\sigma) + E_{kl\sigma\tau\varrho},
\end{aligned} \tag{5.4}
$$

where $E_{kl\sigma\tau\varrho}$ are arbitrary functions of $x^4, \cdots, x^n$. Hence we have the following theorem:

The functions ψ_σ defined by

$$\psi_\sigma = e_\sigma f_\sigma + \sum_\alpha{}' B_{\sigma\alpha} f_\alpha + C_\sigma,$$

where the f_σ are arbitrary functions of $x^\sigma, x^4, \cdots, x^n$, the B's are constants symmetric in their indices and the C's are arbitrary functions of $x^4, \cdots, x^n$, determine a V_n with three fields of parallel vectors; and $E_{kl\sigma\tau\varrho}$ in (5.4) *are arbitrary functions of $x^4, \cdots, x^n$.*

6. **When $p > 3$.** In this case we have from (5.1) equations of the form

$$f_\beta\, \omega_{\tau\alpha\beta} + \varrho_{\tau\alpha\beta} = f_\gamma\, \omega_{\tau\alpha\gamma} + \varrho_{\tau\alpha\gamma} \qquad (\tau, \alpha, \beta, \gamma \neq). \tag{6.1}$$

If this equation be differentiated with respect to x^β and x^γ, the resulting equation leads, by considerations similar to those applied to (3.6), to

$$\frac{\partial \omega_{\tau\alpha\beta}}{\partial x^\gamma} = \frac{\partial f_\gamma}{\partial x^\gamma}\, \omega_{\tau\alpha\beta\gamma}, \qquad \frac{\partial \omega_{\tau\alpha\gamma}}{\partial x^\beta} = \frac{\partial f_\beta}{\partial x^\beta}\, \omega_{\tau\alpha\gamma\beta},$$

where $\omega_{\tau\alpha\beta\gamma}$ is symmetric in β and γ and is independent of $x^\tau, x^\alpha, x^\beta$ and x^γ. From these equations we have

$$\omega_{\tau\alpha\beta} = f_\gamma\, \omega_{\tau\alpha\beta\gamma} + \varrho_{\tau\alpha\beta\gamma}, \qquad \omega_{\tau\alpha\gamma} = f_\beta\, \omega_{\tau\alpha\gamma\beta} + \varrho_{\tau\alpha\gamma\beta},$$

and from (6.1)

$$\varrho_{\tau\alpha\beta} = \varrho_{\tau\alpha\gamma\beta} f_\gamma + \bar{\varrho}_{\tau\alpha\beta\gamma}, \qquad \varrho_{\tau\alpha\gamma} = \varrho_{\tau\alpha\beta\gamma} f_\beta + \bar{\varrho}_{\tau\alpha\gamma\beta}.$$

By continuing this process, we find as the general solution of (3.7)

$$\begin{aligned} F_\sigma = {} & A_\sigma \prod_\alpha{}'_\sigma (f_\alpha) + \sum_{\varrho_1}{}'_\sigma A_{\sigma\varrho_1} \prod_\alpha{}'_{\sigma\varrho_1}(f_\alpha) + \sum_{\varrho_1,\varrho_2}{}'_\sigma A_{\sigma\varrho_1\varrho_2} \prod_\alpha{}'_{\sigma\varrho_1\varrho_2}(f_\alpha) + \cdots \\ & + \sum_{\varrho_1,\cdots,\varrho_{p-2}}{}'_\sigma A_{\sigma\varrho_1\cdots\varrho_{p-2}} \prod_\alpha{}'_{\sigma\varrho_1\cdots\varrho_{p-2}}(f_\alpha) + C_\sigma, \end{aligned} \tag{6.2}$$

where the A's and C's are functions of $x^{p+1}, \cdots, x^n$, the symbol $\sum'_{\sigma}{}_{\varrho_1\cdots\varrho_r}$ means the sum for all different combinations of different ϱ's taking the values $1, \cdots, p$ except σ and only one such term for each combination; and $\prod_\alpha{}'_{\sigma\varrho_1\cdots\varrho_r}$ means the product of the f's except $f_\sigma, f_{\varrho_1}, \cdots, f_{\varrho_r}$. In order that (3.1) be satisfied, we must have $A_1 = A_2 = \cdots = A_p \equiv A$ and the other A's must satisfy the conditions

$$A_{\sigma\varrho_1\cdots\varrho_r} = A_{\tau\varrho_1\cdots\varrho_r} \qquad (\sigma, \tau, \varrho_1, \cdots, \varrho_r \nparallel). \tag{6.3}$$

When these expressions are substituted in (4.2), the latter are reducible to the form $\Phi_\sigma f_\tau + \Psi_\sigma = 0$ where Φ_σ and Ψ_σ are independent of x^τ. Accordingly they must vanish. The first of these conditions is

$$\begin{aligned} \frac{\partial}{\partial x^l}\Big[& A \prod{}'_{\sigma\tau}(f_\alpha) + \sum_{\varrho_1}{}'_{\sigma\tau} A_{\sigma\varrho_1} \prod{}'_{\sigma\tau\varrho_1}(f_\alpha) \\ & + \sum_{\varrho_1,\varrho_2}{}'_{\sigma\tau} A_{\sigma\varrho_1\varrho_2} \prod{}'_{\sigma\tau\varrho_1\varrho_2}(f_\alpha) + \cdots + D_{\sigma\tau}\Big] = 0 \quad (l, \sigma, \tau \nparallel), \end{aligned} \tag{6.4}$$

where

$$D_{\sigma\tau} = A_{\sigma\varrho_1\cdots\varrho_{p-2}} = A_{\tau\varrho_1\cdots\varrho_{p-2}} \qquad (\varrho_1, \cdots, \varrho_{p-2}, \sigma, \tau \nparallel). \tag{6.5}$$

When l takes a value $1, \cdots, p$, different from σ and τ, say ϱ, the above equation becomes

$$\begin{aligned} \frac{\partial f_\varrho}{\partial x^\varrho}\Big[& A \prod{}'_{\varrho\sigma\tau}(f_\alpha) + \sum_{\varrho_1}{}'_{\varrho\sigma\tau} A_{\sigma\varrho_1} \prod{}'_{\varrho\sigma\tau\varrho_1}(f_\alpha) \\ & + \sum_{\varrho_1,\varrho_2}{}'_{\varrho\sigma\tau} A_{\sigma\varrho_1\varrho_2} \prod{}'_{\varrho\sigma\tau\varrho_1\varrho_2}(f_\alpha) + \cdots + E_{\varrho\sigma\tau}\Big] = 0, \end{aligned}$$

where

$$E_{\varrho\sigma\tau} = A_{\sigma\varrho_1\cdots\varrho_{p-3}} = A_{\tau\varrho_1\cdots\varrho_{p-3}} = A_{\varrho\varrho_1\cdots\varrho_{p-3}} \quad (\varrho_1, \cdots, \varrho_{p-3}, \varrho, \sigma, \tau \neq).$$

Since $\partial f_\varrho / \partial x^\varrho \neq 0$, it follows from the above equations that A and all the A's with less than $p-1$ indices are zero. And from (6.4) for $l = p+1, \cdots, n$ we have that the D's (6.5) are constants. Hence from (3.5) and (6.2) we have

$$\psi_\sigma = e_\sigma f_\sigma + \sum_\tau{}'_\sigma D_{\sigma\tau} f_\tau + C_\sigma \quad (\sigma, \tau = 1, \cdots, p;\ \sigma \neq \tau), \tag{6.6}$$

where the D's are constants symmetric in the subscripts and C_σ are arbitrary functions of $x^{p+1}, \cdots, x^n$.

When now we equate the p expressions for each of the functions $g_{kl}\,(k, l = p+1, \cdots, n)$ obtained from (2.6) by taking $\sigma = 1, \cdots, p$, we find that these equations are consistent and that the functions $\varphi_{kl\sigma}$ involve additive arbitrary functions of $x^{p+1}, \cdots, x^n$ as shown in §§ 4, 5. Thus the theorems of these sections can be generalized to any value of $p < n$.

7. When one or more of the functions $\varphi_{\sigma\sigma\tau}$ in (3.2) are zero. If $\varphi_{\sigma\sigma\tau} = 0$ in the second of (3.2), it follows from this equation that $e_\sigma = e_\tau$, if the functions are to be real. From this equation and (3.1) we have then

$$\left(\frac{\partial \psi_\tau}{\partial x^\sigma}\right)^2 = \left(\frac{\partial \psi_\sigma}{\partial x^\sigma}\right)^2, \qquad \left(\frac{\partial \psi_\tau}{\partial x^\tau}\right)^2 = \left(\frac{\partial \psi_\sigma}{\partial x^\tau}\right)^2, \tag{7.1}$$

and from the first of (3.2) we find that $\varphi_{\tau\tau\sigma} = 0$.

If we take $\sigma = 1$, $\tau = 2$, from the first of (7.1) it follows that

$$\psi_2 = \varepsilon\, \psi_1 + \varphi_1, \tag{7.2}$$

where $\varepsilon^2 = 1$ and φ_1 is independent of x^1. Substituting in the second of (7.1), we obtain

$$\left(2\varepsilon \frac{\partial \psi_1}{\partial x^2} + \frac{\partial \varphi_1}{\partial x^2}\right) \frac{\partial \varphi_1}{\partial x^2} = 0.$$

Assume that $\partial \varphi_1 / \partial x^2 \neq 0$; then

$$\psi_1 = \frac{1}{2\varepsilon}(\varphi_2 - \varphi_1) \tag{7.3}$$

where φ_2 is independent of x^2, and from (7.2)

$$\psi_2 = \frac{1}{2}(\varphi_2 + \varphi_1).$$

Substituting in (3.1), we get $\partial \varphi_2 / \partial x^1 = 0$. In this case ψ_1 is independent of x^1 and from (2.5) we get $g_{1l} = 0 \; (l = 1, \cdots, n)$, which is impossible. Hence φ_1 must be independent of x^2, and accordingly we write (7.2) in the form

$$\psi_2 = \varepsilon \psi_1 + \varphi_{12}, \tag{7.4}$$

where φ_{12} is independent of x^1 and x^2.

When the expression (7.4) is substituted in (4.2) for $\sigma = 1$, $\tau = 2$, we get

$$\varepsilon \frac{\partial \psi_1}{\partial x^1} \frac{\partial \varphi_{12}}{\partial x^l} + \varphi_{1l2} = 0 \qquad (l = 3, \cdots, n).$$

Differentiating with respect to x^2, we obtain

$$\frac{\partial^2 \psi_1}{\partial x^1 \partial x^2} \frac{\partial \varphi_{12}}{\partial x^l} = 0.$$

Hence either

$$\psi_1 = f_1(x^1, x^3, \cdots, x^n) + f_2(x^2, x^3, \cdots, x^n), \tag{7.5}$$

or φ_{12} is a constant. In the latter case we have from (2.5) and (7.4)

$$g_{1l} = e_1 \frac{\partial \psi_1}{\partial x^1} \frac{\partial \psi_1}{\partial x^l}, \qquad g_{2l} = e_1 \frac{\partial \psi_1}{\partial x^2} \frac{\partial \psi_1}{\partial x^l} \qquad (l = 1, \cdots, n).$$

Then the fundamental form may be written

$$\varphi = e_1 (d\psi^1)^2 + g_{\alpha\beta}\, dx^\alpha\, dx^\beta \qquad (\alpha, \beta = 3, \cdots, n),$$

that is, the space is a V_{n-1} at most. Hence (7.5) is the only solution.

It is readily found that the two expressions for any $g_{kl} \; (k, l, = 3, \cdots, n)$ and for $\sigma = 1, 2$ from (2.6) are consistent under the above conditions.

From the foregoing considerations it follows that the particular cases when one or more of the functions $\varphi_{\sigma\sigma\tau}$ in (3.2) are zero arise only when constants $D_{\sigma\tau}$ in (6.6) have certain values.

PRINCETON UNIVERSITY,
PRINCETON, N. J.

A SYMBOLIC TREATMENT OF THE GEOMETRY OF HYPERSPACE*

BY

LOUIS INGOLD

Introduction

The beautiful symbolic method of Maschke† for the study of differential invariants has been applied to a certain extent to the study of geometry. Maschke himself has made application to the theory of curvature of hyperspace and also to the study of directional relations.‡ The more important formulas of ordinary differential geometry were developed by Smith§ in terms of the symbolic notations, and finally, Bates|| has used the method in a further study of curvature.

Of these applications, the most complete and satisfactory is that of Maschke in his paper *Differential parameters of the first order*. This paper contains a complete discussion of the relations between the tangent vectors to the subspaces R_k of the space S_n under consideration.

The development of those properties of hyperspace which depend upon invariants and differential parameters of the second order is by no means as complete. Several of the papers mentioned above are, to be sure, devoted to special problems involving second order properties, but no systematic study of these properties by means of the symbolic method has been attempted.

There is, of course, a very extensive development of this subject by means of unsymbolic methods. A quite complete treatment of the geometry of two-dimensional surfaces has been given by Wilson and Moore¶ and

* Presented to the Society (Southwestern Section), December 1, 1923.

† Maschke, *A new method of determining the differential parameters and invariants of quadratic differential quantics*, these Transactions, vol. 1 (1900), pp. 197–204, and *A symbolic treatment of the theory of invariants of quadratic differential quantics in n variables*, ibid., vol. 4 (1903), pp. 445–469. This paper will be referred to as *Invariants*.

‡ Maschke, *The Kronecker-Gaussian curvature of hyperspace*, and *Differential parameters of the first order*, these Transactions, vol. 7 (1906), pp. 69–80, and pp. 81–93.

§ A. W. Smith, *The symbolic treatment of differential geometry*, these Transactions, vol. 7 (1906), pp. 33–60.

|| Bates, *An application of symbolic methods to the treatment of mean curvatures in hyperspace*, these Transactions, vol. 12 (1911), pp. 19–38.

¶ E. B. Wilson and C. L. E. Moore, *Differential geometry of two-dimensional surfaces in hyperspace*, Proceedings of the American Academy of Arts and Sciences, vol. 52 (1916), p. 269.

Moore* has extended some of their results to spaces of n dimensions. These authors make use of the method of the absolute differential calculus,† and this method has been employed in the majority of recent papers on this subject.

It is the object of this paper to study the second order properties of n-dimensional manifolds by means of the symbolic method of Maschke. It is found that many important results and formulas of the geometry of hyperspace are easily derived by this method. One advantage of the method is the ease with which relations may be derived connecting various invariants and differential parameters.

An advantage of the method of the absolute calculus which is always, very justly, insisted upon is that derived from the fact that invariant expressions are immediately recognizable on account of their form. This applies with equal force to the symbolic method. It may be added that a general vector interpretation of the Maschke symbol makes it possible to study the properties of hyperspace without reference to a containing euclidean space.

Incidentally the paper contains a number of extensions of the symbolic theory: for example, a formula is found which expresses as an invariant, or differential parameter, the derivative of any function of the coördinates with respect to arc length along any curve.

By means of this formula it is possible not only to obtain a large number of invariants but also, from known relations connecting differential parameters, to obtain still other relations. The formula also aids very materially in the geometric interpretation of invariant forms.

In order to avoid frequent interruptions in the geometric applications a general account of the notations and interpretations of the symbolic theory has been given in Part I. This part also includes certain general formulas that are of frequent occurrence in the latter portions of the paper. Part II treats the geometry of two-dimensional manifolds in considerable detail. In Part III is given an outline of the extension to n dimensions.

I. Notations and general formulas

1. **First fundamental form.** If the space S_n under consideration is contained in a euclidean space we may let f denote the vector from an arbitrary origin to the points of S_n. The square of the differential of arc length along any curve of S_n is then given by

$$(1)\qquad ds^2 = (\textstyle\sum f_i\, du_i)^2 = \sum f_i f_j\, du_i\, du_j = \sum E_{ij}\, du_i\, du_j,$$

* C. L. E. Moore, *Note on minimal varieties in hyperspace*, Bulletin of the American Mathematical Society, vol. 27 (1921), p. 216.

† For references see the paper by Wilson and Moore, loc. cit.

where the multiplication of the vectors is the inner or scalar multiplication, and the subscript i denotes differentiation with respect to the corresponding variable u_i.*

The scalar products $f_i f_j$ are the coefficients of the differential form (1). The vector f, therefore, has precisely the properties of Maschke's symbolic function belonging to (1).†

It is not necessary, however, to assume that S_n is contained in a euclidean space. The developments of the paper apply to spaces S_n defined, or characterized, by a function $f(x; u_1, u_2, \cdots, u_n)$ which is such that

$$\int_a^b f_i f_j \, dx = E_{ij}.$$

There are, of course, functions of this sort that are not expressible by means of a finite number of terms in the form

$$\sum_i U_i \, e_i(x),$$

where the coefficients U_i are functions of the parameters u_i only. For many such interpretations of the symbol f there can be obtained a theory wholly analogous to the geometric theory.

In order to form expressions of higher than the first degree in the coefficients E_{ij} Maschke introduced other symbols φ, ψ, or $f^1, f^2, \cdots, \varphi^1, \varphi^2, \cdots$, equivalent to the symbol f. (Upper indices will be frequently used instead of different letters to indicate distinct functions or symbols. Exponents are seldom used and when used will be easily recognized.) For the purposes of this paper, these will be regarded as equivalent notations for the vector f; thus, $f_1 f_2 \varphi_1 \varphi_2 = E_{12}^2$ and $f_1 f_2 \varphi_2 \varphi_3 \psi_1 \psi_3 = E_{12} E_{23} E_{31}$.

Those symbols (or notations) that occur precisely twice in a symbolic product are to be combined as scalar products. If there is only one symbol that occurs just once, the expression represents a simple vector. The same symbol (or notation) never occurs more than twice as a factor. It is very useful to notice that if each of two distinct symbols occurs precisely twice as a factor, then the two symbols may be interchanged without affecting the value of the product.

---

* The subscript is generally used in this sense, but occasionally it is used merely as a distinguishing mark, as for example in the fundamental quantities E_{ij}.

† For details relating to the vector interpretation of symbolic forms see the author's paper *Vector interpretation of symbolic differential parameters*, these Transactions, vol. 11 (1910), pp. 449–474; also *Functional differential geometry*, ibid., vol. 13 (1912), pp. 318–341.

Maschke has shown that if $F^1, F^2, \cdots, F^n$ are n arbitrary functions of the variables $u_1, \cdots, u_n$, or any invariants of the form (1), then β times the jacobian of $F^1, \cdots, F^n$ is again an invariant of (1), where β denotes the reciprocal square root of the determinant $|E_{ij}|$. He calls this product

$$\beta\{F^1, F^2, \cdots, F^n\}$$

an invariantive constituent of the form (1) and denotes it by $(F^1, F^2, \cdots, F^n)$ or more briefly by (F).*

2. **Vector invariants of the first order.** In the invariantive constituent just described some or all of the F's may be symbols or other invariantive constituents. If k distinct symbols occur just once the invariantive constituent represents a k-dimensional vector; thus $(\varphi\, a) = (\varphi^1, \cdots, \varphi^k, a^1, \cdots, a^{n-k})$† represents a k-dimensional vector tangent to the k-dimensional subspace determined by the equations $a =$ const., $a =$ const., $\cdots$, $a =$ const. This vector is clearly invariant under change of parameters.

Other important invariants may be obtained by multiplying $(\varphi\, a)$ by a constituent which contains one or more of the notations φ together with other symbols or scalar functions. In this way is formed the invariant vector $(f\varphi)(\varphi\, a) = (f^1 \cdots f^{n-k}\,\varphi^1 \cdots \varphi^k)(\varphi^1 \cdots \varphi^k\, a^1 \cdots a^{n-k})$ which represents an $(n-k)$-dimensional vector tangent to S_n but normal to the space determined by the a^i as indicated above.

In some cases the product of several invariantive constituents may be interpreted in more than one way; thus the identity

$$(f V^1 \cdots V^{r-1}\,\varphi^1 \cdots \varphi^{n-r})\,(u^1 \cdots u^r\,\varphi^1 \cdots \varphi^{n-r})\,(f u^1 \cdots u^r\, w^1 \cdots w^{n-r}) = 0$$

may be interpreted as stating that the vector $(fV\varphi)(u\varphi)$ is orthogonal to the vector (fuw) or that the two $(n-r)$-dimensional vectors $(fV\varphi)(fuw)$ and $(u\varphi)$ are orthogonal to each other. In general the expression

$$(f V^1 \cdots V^{r-1}\,\varphi^1 \cdots \varphi^{n-r})\,(f a^1 \cdots a^{n-1})$$

represents an $(n-r)$-dimensional vector tangent to S_n but normal to the subspace determined by the equations

* Maschke, (*Invariants*, loc. cit., p. 447) uses the brackets { } to denote the jacobian of the quantities enclosed and reserves the parenthesis ( ) for β times the jacobian.

† Maschke writes this in the form $(\varphi^1, \cdots, \varphi^k, a)$ where a represents $a^1, a^2, \cdots, a^{n-k}$. See *Invariants*, loc. cit., p. 448,. Where no misunderstanding is likely to occur this may be further condensed to $(\varphi\, a)$.

$$a^{i_1} = \text{const.},\ a^{i_2} = \text{const.}, \cdots, a^{i_r} = \text{const.},$$

where $i_1, i_2, \cdots, i_r$ is any combination of r of the indices $1, 2, \cdots, n-1$.

3. **Vector operators.** It is sometimes convenient to regard symbolic invariantive constituents as vector operators; thus in any of the products of the preceding article either factor could be regarded as an operator applied to the other factor. The operator $(f^1 \cdots f^k \varphi^1 \cdots \varphi^{n-k})$ applied to the k-dimensional vector $(f^1 \cdots f^k a^1 \cdots a^{n-k})$ converts it into an $(n-k)$-dimensional vector in a space normal to the operand. This operator is thus analogous to Grassmann's operation of taking the supplement of a vector.

There are linear combinations of invariantive constituents which cannot be regarded as simple vectors, but these can often be regarded as compound elements in Grassmann's sense, or they may be regarded as tensors; for example, the sum $(f\varphi a b) + (f\varphi c d)$ is not a simple two-dimensional vector, but a compound element (zusammengesetzte Grösse). If this sum, however, is multiplied by a one-dimensional vector in either the f notation or the φ notation the result is a one-dimensional vector. Compound elements, as well as simple elements, can be used as operators to convert vectors or operators into other vectors or operators.

4. **Vectors and vector invariants of the second order.** The forms mentioned so far have involved only the first derivatives of the vector f. We call them vectors or invariants of the first order. Vectors or vector operators which involve second derivatives of f (or any of its equivalents) will be called vectors or vector operators of the second order.

The vectors f_{ij} are as a rule not tangent to the space defined by f; thus if this space is contained in a euclidean space of r dimensions we may write

$$f = \sum_{i=1}^{i=r} x^i e^i$$

where the e's are independent vectors of the containing space and the x's are functions of the parameters u of the space defined by f. The f_{ij} will then certainly lie in this larger space.

It is evident that the f_{ij} are not completely defined by the differential form $\sum E_{ij}\, du_i\, du_j$; in fact f itself is not completely determined, since it may be the defining vector of any of the surfaces applicable to one another each having the same first differential form.

There are, however, certain restrictions on the vectors f_{ij}. From the equation $f_i f_j = E_{ij}$, we obtain by differentiation $f_i f_{jk} + f_{ik} f_j = \partial E_{ij} / \partial u_k$ and from this we obtain Maschke's formulas for the Christoffel triple index symbols

$$f_i f_{km} = \frac{1}{2}\left[\frac{\partial E_{ik}}{\partial u_m} + \frac{\partial E_{im}}{\partial u_k} - \frac{\partial E_{km}}{\partial u_i}\right].$$

Thus the scalar products of the tangent vectors f_i and the vectors f_{km} are completely determined by the differential form (1). Maschke has shown that the combinations $f_{ir} f_{ks} - f_{kr} f_{is}$ are also completely determined. For ordinary space the various restrictions on the second derivatives of the defining vector f are all summed in the well known relations of Gauss and Codazzi. The corresponding relations for spaces in general will be given later.

The tangent n-dimensional vector $(f^1, \cdots, f^n)$, where $f^1, \cdots, f^n$ are all equivalent symbols of the differential form (1), is an invariant of the first order. The derivatives of this vector with respect to the parameters u_i will all be expressions of the second order. They are not, however, in general simple elements but compound elements in a space of more than n dimensions.

5. **Normal vectors.** Although the vectors f_{ij} do not, in general, lie in the tangent space to the space f they are, as a rule, not normal. It can be shown, however, that the first normals to all curves lying in f are expressible in terms of the tangent vectors f_i and the vectors f_{ij}. Any normal to the surface that can be expressed in terms of the first and second derivatives of f will be called a first normal. If the f_{ij} are linearly independent there are $n(n+1)/2$ linearly independent first normals. One possible choice of these is the set

$$f_{ij} - \sum \left\{ \begin{matrix} i\ j \\ k \end{matrix} \right\} f_k = \mathbf{N}_{ij} \tag{2}$$

where $\left\{ \begin{matrix} i\ j \\ k \end{matrix} \right\}$ are Christoffel's triple index symbols of the second kind belonging to the differential form (1).

Other normal vectors to a two-dimensional surface f can be obtained from the forms $(f\varphi)_1$ and $(f\varphi)_2$. These forms, as previously mentioned, are compound elements but when applied as operators to the vectors $\boldsymbol{\varphi}_1$ and $\boldsymbol{\varphi}_2$ they give expressions which are linear in the f_{ij} and the f_k. We write

$$\mathbf{a} = \varphi_1(f\varphi)_1, \quad \mathbf{b} = \varphi_2(f\varphi)_2, \quad \mathbf{c} = \varphi_1(f\varphi)_2, \quad \mathbf{d} = \varphi_2(f\varphi)_1.^* \tag{3}$$

It is easily verified that each of these vectors is orthogonal to each of the tangent vectors f_1 and f_2.

The following vectors, written in invariant form, are all normal to the

* Black face type will be used for letters indicating vectors except for the vector f and its derivatives in any of its equivalent notations.

surface f because they are linearly expressible in terms of the vectors $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d}$ given above:

$$(\varphi a)((f\varphi)a), \qquad (\psi\varphi)(\psi a)((f\varphi)a),$$
$$(\varphi a)(\psi a)((f\varphi)\psi), \qquad (\psi\varphi)(\psi(f\varphi)).$$

The last of these vanishes identically* and this gives the linear relation $Eb - F(c+d) + Ga = 0$ which connects the normal vectors $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d}$.

In n dimensions we have the formula $(f, \varphi^1, \cdots, \varphi^{n-1})^2 = n!$, or, more briefly, $(f\varphi)^2 = n!$; and consequently $(f\varphi)(f\varphi)_i = 0$.

This shows that the compound n-dimensional vectors $(f\varphi)_i$ are orthogonal to the n-dimensional (simple) tangent vector $(f\varphi)$.

As in the two-dimensional case we have the normal vectors

$$a_{ij} = \Phi_i(f\varphi)_j,$$

where Φ_i is the cofactor of f_i in the determinant

$$\{f, \varphi^1 \cdots \varphi^{n-1}\}.$$

That these are really orthogonal to the tangent vectors is easily seen by means of Maschke's formula†

$$f_k(f\varphi)_i(a\varphi) = 0,$$

where a is an arbitrary function; for example, if a is taken to be the independent parameter u_j, $(a\varphi)$ reduces to $\beta\varphi_j$ and the formula shows that $\beta f_k \cdot a_{ji} = 0$.

Maschke's formula (52) referred to in the foot note below may be written in the form

$$(\psi\varphi)(\psi, (f\varphi), a^3, \cdots, a^n) = 0,$$

where $a^3, \cdots, a^n$ are arbitrary functions of the parameters u.

6. The equations of Gauss and Codazzi. The scalar products of the first derivatives of the vector f are the first fundamental quantities. Similarly the scalar products of the second derivatives could be taken as the second fundamental quantities. It seems more convenient, however, to use scalar products of the normal vectors as defined in equations (2), p. 579. We write $N_{ij} N_{rs} = L_{ijrs}$.

---

* See Maschke, *Invariants*, loc. cit., p. 455, formula (52).

† *Invariants*, loc. cit., formula (33).

The equations (2) may be written

$$
(4) \qquad \begin{aligned} f_{ij} &= \sum_r \begin{Bmatrix} i\,j \\ r \end{Bmatrix} f_r + N_{ij}, \\ f_{ik} &= \sum_r \begin{Bmatrix} i\,k \\ r \end{Bmatrix} f_r + N_{ik}. \end{aligned}
$$

Differentiating the first with respect to u_k and the second with respect to u_j and equating the results we obtain

$$
(5) \qquad \sum_r \begin{Bmatrix} i\,j \\ r \end{Bmatrix} f_{rk} + \sum_r \begin{Bmatrix} i\,j \\ r \end{Bmatrix}_k f_r + [N_{ij}]_k = \sum_r \begin{Bmatrix} i\,k \\ r \end{Bmatrix} f_{rj} + \sum_r \begin{Bmatrix} i\,k \\ r \end{Bmatrix}_j f_r + [N_{ik}]_j.
$$

From this equation we obtain, after multiplying by f_m and transposing,

$$
(6) \qquad \begin{aligned} \sum_r \begin{Bmatrix} i\,j \\ r \end{Bmatrix} \begin{bmatrix} r\,k \\ m \end{bmatrix} - \sum_r \begin{Bmatrix} i\,k \\ r \end{Bmatrix} \begin{bmatrix} r\,j \\ m \end{bmatrix} + \sum_r E_{rm} \left[\begin{Bmatrix} i\,j \\ r \end{Bmatrix}_k - \begin{Bmatrix} i\,k \\ r \end{Bmatrix}_j \right] \\ = f_m [N_{ik}]_j - f_m [N_{ij}]_k. \end{aligned}
$$

Now if the first of equations (4) is multiplied by N_{mk} it is found that $N_{mk}\,N_{ij} = N_{mk} f_{ij} = L_{ijmk}$; also from the equation $N_{ij} f_m = 0$ we obtain by differentiation

$$
N_{ij} f_{mk} + f_m [N_{ij}]_k = 0.
$$

Using these equations the right hand side of (6) reduces to

$$
L_{ijmk} - L_{ikmj}.
$$

Equation (6) then shows that the difference $L_{ijmk} - L_{ikmj}$ is a function of the first fundamental quantities and their derivatives. This is the generalization of Gauss's equation.

The second fundamental quantities L must also satisfy a second set of equations which are the generalization of the Codazzi equations of ordinary surface theory. These equations, however, involve still other quantities besides the first and second fundamental quantities. The equations are obtained by multiplying both sides of (5) by the vector N_{ms}. The products $N_{ms}\,N_{ijk}$ will be denoted by $\begin{pmatrix} i\,j\,k \\ m\,s \end{pmatrix}$. We obtain, then,

$$
\sum_r \begin{Bmatrix} i\,j \\ r \end{Bmatrix} L_{rkms} - \sum_r \begin{Bmatrix} i\,k \\ r \end{Bmatrix} L_{rjms} = \begin{pmatrix} i\,k\,j \\ m\,s \end{pmatrix} - \begin{pmatrix} i\,j\,k \\ m\,s \end{pmatrix}.
$$

The difference $\binom{ikj}{ms} - \binom{ijk}{ms}$ cannot be expressed independently of the relations just found, in terms of the L's and the E_{ij}, without further limitation on the space f. If, for example, the discussion is limited to spaces which have a single first normal the relations reduce to the usual Codazzi equations. Other forms of limitation on f will lead to different forms of these integrability conditions.

7. Resolution of vectors in given directions. A tangent vector is resolved into its components in n given directions tangent to the space in the usual manner; thus in a two-dimensional space consider the tangent vector

$$(f\varphi)((\varphi a)a).$$

To resolve this vector along the tangents to the curves $b=$ const. and $c=$ const. we write

$$(f\varphi)((\varphi a)a) = l(fb)+m(fc). \tag{7}$$

By multiplying in turn by (fb) and (fc) two equations are obtained from which l and m can be determined:

$$\begin{aligned}(fb)(f\varphi)((\varphi a)a) &= l(fb)^2+m(fb)(fc),\\ (fc)(f\varphi)((\varphi a)a) &= l(fb)(fc)+m(fc)^2.\end{aligned}$$

If the determinant of the coefficients of l and m in these equations is zero the two directions (fb) and (fc) are not independent. But when this is not the case the equations can be solved for l and m.

The values of l and m are evidently differential parameters and their geometric meanings are easily seen from the manner in which they are derived.

If (fb) and (fc) are mutually orthogonal, the expressions for l and m become quite simple since, in this case, $(fb)(fc)=0$. The expression for $(f\varphi)((\varphi a)a)$ then becomes

$$(f\varphi)((\varphi a)a) = \frac{(\psi b)(\psi\varphi)((\varphi a)a)}{(\psi b)^2}(fb)+\frac{(\psi c)(\psi\varphi)((\varphi a)a)}{(\psi c)^2}(fc). \tag{8}$$

These results and formulas are easily extended to n dimensions.

Another method which will frequently be used is the following. Let (fab) be a tangent vector in a three-dimensional space.* The product $(fab)(pqr)$, where p, q, and r are scalar functions, has the same direction as (fab). This product can be written in several different ways as a sum

* The method may be extended in an obvious way to vectors in n dimensions.

of three such products. Any one element of either factor may be exchanged with each element of the other factor. The sum of the resulting products is equivalent to the original product; thus,

$$\text{(A)} \qquad (fab)(pqr) = (fap)(bqr) + (fpb)(aqr) + (pab)(fqr).$$

In this way, the vector (fab) is resolved into three components along (fap), (fpb), and (fqr). In case the functions a, b, p, q, r are not independent in sets of three, some of the terms on the right vanish and no resolution of the vector (fab) is effected; for example, if a is a multiple of p the first and last terms vanish and the remaining one is a multiple of (fab).

Still another method which is sometimes useful for second order vectors in two dimensions depends on the identity

$$\text{(B)} \qquad ((ab)c) + ((bc)a) + ((ca)b) = 0.$$

By means of this the vector $((fa)b)$ can be expressed in terms of the vectors $((fb)a)$ and $((ab)f)$. This is not immediately applicable to higher dimensions but the other methods mentioned in this article can be used in any number of dimensions.

8. **Derivatives with respect to arc length.** In all the work that follows it will be convenient to be able to write down in invariant form the derivative of any function of the coördinates of a given space with respect to arc length along any curve.

Consider first an arbitrary two-dimensional surface defined by the vector $f(u, v)$ and let $a =$ const. define a curve on this surface.

Let V be any vector or scalar function of u and v.

For values of u and v along the curve a, V is a function of a single variable, since u and v satisfy the equation $a =$ const.

Denoting arc length by s we have

$$\text{(9)} \qquad \frac{dV}{ds} = V_1 \frac{du}{ds} + V_2 \frac{dv}{ds}.$$

But

$$a_1 \frac{du}{ds} + a_2 \frac{dv}{ds} = 0,$$

and

$$\frac{du}{ds} = \frac{du}{\sqrt{(E du^2 + 2F du\, dv + G dv)^2}} = \frac{a_2}{\sqrt{(E a_2^2 - 2F a_1 a_2 + G a_1^2)}} = \frac{\beta a_2}{\sqrt{\Delta_1 a}},$$

where $\Delta_1 a$ is the well known first differential parameter.

Hence, by substitution in (9) we obtain

$$\frac{dV}{ds} = \frac{1}{a_1}\frac{du}{ds}[V_1 a_2 - V_2 a_1] = \frac{(Va)}{\sqrt{\Delta_1 a}}. \tag{10}$$

Thus, whether V represents a vector or a scalar function of u and v we obtain dV/ds by simply forming the invariant $(Va)/\sqrt{\Delta_1 a}$.

This result can be generalized in the following way. Consider a space f of n dimensions and let $\mathfrak{r} = \sum a^i f_i$ be any vector function of the coördinates tangent to f. The length of this vector is $\sqrt{\sum E_{ij} a^i a^j} = \sqrt{\mathfrak{r}^2}$.

If V is any function of the coördinates we have for the derivative of V with respect to arc length in any direction

$$\frac{dV}{ds} = \sum V_i \frac{du_i}{ds}.$$

But for any curve tangent to the vector $\mathfrak{r}$

$$\frac{du_i}{ds} = \lambda a_i,$$

where λ is a proportionality factor, the same for every i. This factor is obtained from the equation

$$\sum E_{ij}\frac{du_i}{ds}\frac{du_j}{ds} = 1 = \lambda^2 \sum E_{ij} a^i a^j,$$

whence

$$\lambda^2 = \frac{1}{\mathfrak{r}^2}.$$

If we substitute these results in the expression for dV/ds we obtain

$$\frac{dV}{ds} = \frac{\sum a^i (\partial V/\partial u_i)}{\sqrt{\sum E_{ij} a^i a^j}}.$$

It follows that if $\mathfrak{r}$ is any vector tangent to the space f and expressed in the f notation, the derivative of any function V with respect to the arc length in the direction $\mathfrak{r}$ is obtained from the expression for $\mathfrak{r}$ by replacing f_i by V_i $(i = 1, 2, \cdots, n)$ and dividing the result by the length of $\mathfrak{r}$.

As an example we write down the value of $dV/d\sigma$ where σ denotes arc length along an orthogonal trajectory of $a =$ const. on a two-dimensional surface. In this case $\mathfrak{r} = (f\varphi)(\varphi a)$ and

$$\frac{dV}{d\sigma} = \frac{(V\varphi)(\varphi a)}{\sqrt{\Delta_1 a}}.$$

In applications the denominator in these formulas can often be omitted. For example from the equation

$$(fa)(fb) = 0$$

which holds for two orthogonal families of curves in a two-dimensional manifold, we obtain by "differentiation"

$$(fa)((fb)c) + ((fa)c)(fb) = 0.$$

Here the differentiation is performed with respect to arc length along the curve $c =$ const. and the denominator of the formula is omitted.

II. Two-dimensional surfaces

9. **Sequence of curvatures for any curve on a surface.** We may apply the formulas given in the paper *Functional differential geometry** for the normals and curvatures of any curve to find expressions for the normals and curvatures of any curve lying in any space.

For curves lying in a two-dimensional surface we have

$$\begin{aligned} n_0 &= t = \frac{(fa)}{\sqrt{(fa)^2}}, \\ \frac{n_1}{r_1} &= \frac{(ta)}{\sqrt{(fa)^2}}, \\ &\cdots\cdots\cdots\cdots \\ \frac{n_j}{r_j} &= \frac{(n_{j-1}a)}{\sqrt{(fa)^2}} + \frac{n_{j-2}}{r_{j-1}}, \end{aligned} \tag{11}$$

where $a =$ const. is the equation of the curve.

When these results are expressed in terms of the vector f and its derivatives they furnish a perfectly definite method for obtaining in succession the required normals and curvatures.

For the principal normal (first normal), we have

$$\frac{n_1}{r_1} = \frac{((fa)a)}{(\varphi a)^2} - \frac{(\varphi a)((\varphi a)a)}{[(\varphi a)^2]^2}(fa), \tag{12}$$

and from this we obtain

$$\frac{1}{r_1^2} = \frac{-[(\varphi a)((\varphi a)a)]^2 + (\varphi a)^2((fa)a)^2}{[(\varphi a)^2]^3}.$$

* Loc. cit., p. 321.

This formula for the square of the first curvature of any curve on the surface f contains in the denominator and in the first term of the numerator only the quantities a, E, F, G, and their derivatives. In the second term of the numerator, however, there occur certain of the products $f_{ij}\,f_{rs}$ which are second order quantities not expressible in terms of the E, F, G. These we regard as known.

If the expression for $\boldsymbol{n}_1$ is substituted in the formula for $\boldsymbol{n}_2/r_2$ it is seen that

$$\boldsymbol{n}_2/r_2 = p(fa) + q((fa)\,a) + r(((fa)\,a)\,a),$$

where p, q, and r are certain differential parameters. In like manner we have in general

$$\boldsymbol{n}_j/r_j = p_1(fa) + p_2((fa)\,a) + \cdots + p_{j+1}(\cdots(fa)\cdots a),$$

where in the last term there are $j+1$ parentheses.

It is clear that the square of the second curvature will contain third order quantities, the square of the third curvature fourth order quantities, and so on.

10. **Components of the curvature vectors.** The tangent vector (fa) can be written in several different forms, and from each of them the curvature vector of the curve $a=$ const. can be computed. This gives a variety of different expressions for the curvature vector.

If we obtain $d\boldsymbol{t}/ds$ from $(f\varphi)(\psi\varphi)(\psi a) = \boldsymbol{t}\sqrt{(fa)^2}$, one of the terms of the result is normal to the surface and the remaining terms are tangent to the surface. Hence we have

$$\frac{d\boldsymbol{t}}{ds} = \frac{(\psi\varphi)(\psi a)}{\sqrt{(fa)^2}}\,\frac{((f\varphi)\,a)}{\sqrt{(fa)^2}} + \text{tangent vector}.$$

The tangential component is also easily obtained from this form of $\boldsymbol{t}$, but it is more easily simplified when derived from the form on the right of equation (12). We write

$$((fa)\,a) = p(fa) + q(f\varphi)(\varphi a) + \boldsymbol{N}.$$

The second term is tangent to the surface but normal to the curve. Since (fa) and $(f\varphi)(\varphi a)$ are orthogonal to each other we may multiply in turn by (fa) and $(f\varphi)(\varphi a)$ and obtain

$$p = \frac{((fa)\,a)(fa)}{(fa)^2}, \qquad q = \frac{((fa)\,a)(f\varphi)(\varphi a)}{(fa)^2}.$$

Thus the component of $((fa)a)/(fa)^2$ along the tangent vector (fa) just cancels the second term on the right in equation (12). The curvature vector then takes the form

$$(13)\qquad \frac{dt}{ds}=\frac{(\psi\varphi)(\varphi a)((\psi a)a)}{[(fa)^2]^2}(f\theta)(\theta a)+\frac{(\psi\varphi)(\psi a)((f\varphi)a)}{(fa)^2}.$$

It is evident that there should be no component in the direction of (fa) since if the first curvature vector of the curve $a=$ const. is resolved into two components, one normal to the surface, and the other tangent to the surface, the tangential component is orthogonal to the curve.

If we denote the vector of unit length in the direction of $(f\varphi)(\varphi a)$ by $\mathbf{t}$ and the normal component of dt/ds by $\mathfrak{a}$, equation (13) becomes

$$(14)\qquad \frac{dt}{ds}=\Gamma_1 a\,\mathbf{t}+\mathfrak{a},$$

where $\Gamma_1 a$ is the important differential parameter

$$\frac{(\psi\varphi)(\varphi a)((\psi a)a)}{[\sqrt{(fa)^2}]^3}.$$

11. **Other curvature vectors.** Besides the vector dt/ds already studied, there are three other important curvature vectors: $d\mathbf{t}/d\sigma$, $dt/d\sigma$, and $d\mathbf{t}/ds$, where σ denotes arc length along the orthogonal trajectories of the curves $a=$ const.

Proceeding as above for dt/ds we obtain, after reduction, the following formulas:

$$(15)\qquad \begin{aligned}
\frac{dt}{ds}&=\frac{(\psi\varphi)(\varphi a)((\psi a)a)\mathbf{t}}{[\Delta_1 a]^{3/2}}+\frac{(\psi\varphi)(\psi a)((f\varphi)a)}{[\Delta_1 a]},\\
\frac{d\mathbf{t}}{d\sigma}&=\frac{(f\varphi)(fa)(\theta a)((\varphi a)\theta)t}{[\Delta_1 a]^{3/2}}+\frac{(\varphi a)(\theta a)((f\varphi)a)}{[\Delta_1 a]},\\
\frac{dt}{d\sigma}&=\frac{(\psi\varphi)(\varphi a)(\theta a)((\psi a)a)\mathbf{t}}{[\Delta_1 a]^{3/2}}+\frac{(\psi\varphi)(\psi a)(\theta a)((f\varphi)\theta)}{[\Delta_1 a]},\\
\frac{d\mathbf{t}}{ds}&=\frac{(\varphi\psi)(\varphi a)((\psi a)a)t}{[\Delta_1 a]^{3/2}}+\frac{(\varphi a)((f\varphi)a)}{[\Delta_1 a]}.
\end{aligned}$$

The formula for dt/ds is rewritten here and $\Delta_1 a$ is used instead of $(fa)^2$. This is the well known differential parameter of the first order of ordinary differential geometry.

The numerator of the expression for the normal component of $dt/d\sigma$ can be simplified by resolving it into components according to rule A of

§ 7. In the products $((f\varphi)\theta)(\psi a)$ exchange the element θ with the elements ψ and a. Introducing the other two factors, the result may be written

$$\begin{aligned}(\psi\varphi)(\theta a)((f\varphi)\theta)(\psi a) &= (\psi\varphi)(\theta a)[((f\varphi)\psi)(\theta a)+((f\varphi)a)(\psi\theta)] \\ &= (\psi\varphi)((f\varphi)\psi)(\theta a)^2+(\psi\varphi)(\psi\theta)(\theta a)((f\varphi)a).\end{aligned}$$

The first of these terms vanishes identically and the other reduces to $(\varphi a)((f\varphi)a)$. It follows that the normal components of $d\mathbf{t}/d\sigma$ and $d\mathbf{t}/ds$ are the same.

Equations (15) can now be rewritten as follows:

$$(16) \qquad \begin{aligned} \frac{dt}{ds} &= \Gamma_1 a\,\mathbf{t}+\mathfrak{a}; & \frac{dt}{d\sigma} &= -\Gamma_2 a\,\mathbf{t}+\mathfrak{m}; \\ \frac{d\mathbf{t}}{d\sigma} &= \Gamma_2 a\,t+\mathfrak{b}; & \frac{dt}{ds} &= \Gamma_1 a\,t+\mathfrak{m}, \end{aligned}$$

where $\mathfrak{a}$ and $\mathfrak{b}$ denote the normal components of the curvature vectors of the curve $a =$ const. and its orthogonal trajectory and $\mathfrak{m}$ denotes the normal component of either of the cross curvatures.

The differential parameters

$$\Gamma_1 a = \frac{(\psi\varphi)(\varphi a)((\psi a)a)}{[\Delta_1 a]}, \qquad \Gamma_2 a = \frac{(f\varphi)(fa)(\theta a)((\varphi a)\theta)}{[\Delta_1 a]}$$

are the geodesic curvatures of the curves $a =$ const. and of their orthogonal trajectories. The curves $a =$ const. are geodesics if $(\psi\varphi)(\varphi a)((\psi a)a) = 0$.

12. **Relations connecting normal vectors.** The three normals $\mathfrak{a}, \mathfrak{b}, \mathfrak{m}$ are invariant under change of parameters. They depend, however, on the function a.

From the two vectors $\mathfrak{a}$ and $\mathfrak{b}$ a vector can be obtained which is an invariant proper; i. e. one that does not depend upon an arbitrary function a.

From the identity

$$(\psi\varphi)((f\varphi)a)+((f\varphi)\psi)(\varphi a) = ((f\varphi)\varphi)(\psi a)$$

we have immediately

$$(17) \qquad \mathfrak{a}+\mathfrak{b} = \frac{((f\varphi)\varphi)(\psi a)^2}{\Delta_1 a} = ((f\varphi)\varphi),$$

and this expression does not contain the function a. It follows that the sum of the normal components of the first curvature vectors of any pair of orthogonal curves through a point is the same as the corresponding sum for any other pair. The invariant vector $\mathfrak{a}+\mathfrak{b}$ will be denoted by $2\boldsymbol{h}$.

We may also determine the connection between the normal components of the curvature vectors of two different systems of curves.

Any vector function tangent to the surface may be written in the form $p\mathbf{t}+q\mathfrak{t}$ where p and q are functions of the coördinates. If $p^2+q^2=1$ the length of the corresponding vector at each point of the surface is unity.

The curvature vector of the curves defined by the vector $p\mathbf{t}+q\mathfrak{t}$ $(p^2+q^2=1)$ may be written

$$p\frac{(p\mathbf{t}+q\mathfrak{t},\, a)}{\sqrt{(fa)^2}}+q\frac{(p\mathbf{t}+q\mathfrak{t},\, \varphi)(\varphi a)}{\sqrt{(fa)^2}},$$

or also in the form

$$p\frac{d[p\mathbf{t}+q\mathfrak{t}]}{ds_a}+q\frac{d[p\mathbf{t}+q\mathfrak{t}]}{d\sigma_a},$$

where s_a denotes arc length along $a=$ const. and σ_a means arc length along the orthogonal trajectories of the curves a.

When expanded the values of $\frac{d\mathbf{t}}{ds_a}$, $\frac{d\mathfrak{t}}{ds_a}$, $\frac{d\mathbf{t}}{d\sigma_a}$, and $\frac{d\mathfrak{t}}{d\sigma_a}$ from equations (16) may be substituted. The result reduces to

$$\text{(18)}\qquad \begin{aligned}\frac{d[p\mathbf{t}+q\mathfrak{t}]}{ds} = {} & \left[-\frac{(qa)}{\sqrt{(fa)^2}}+\frac{(p\varphi)(\varphi a)}{\sqrt{(fa)^2}}-p\Gamma_1+q\Gamma_2\right](q\mathbf{t}-p\mathfrak{t}) \\ & +p^2\mathfrak{a}+2pq\mathfrak{m}+q^2\mathfrak{b}.\end{aligned}$$

It is clear from this equation that the tangential component of the curvature of the curves tangent to $p\mathbf{t}+q\mathfrak{t}$ has at each point the direction of the orthogonal vector $q\mathbf{t}-p\mathfrak{t}$. The magnitude of this component, given by the coefficient in brackets, is the geodesic curvature of the system of curves determined by $p\mathbf{t}+q\mathfrak{t}$.

The normal component is $\bar{\mathfrak{a}}=p^2\mathfrak{a}+2pq\mathfrak{m}+q^2\mathfrak{b}$. If we write $p=\cos\theta$, $q=\sin\theta$, this becomes

$$\mathfrak{a}\cos^2\theta+2\,\mathfrak{m}\sin\theta\cos\theta+\mathfrak{b}\sin^2\theta.$$

The derivative of the orthogonal unit vector, $q\mathbf{t}-p\mathfrak{t}$, may also be computed; likewise the two cross curvatures. The results are

$$\frac{d[q\mathbf{t}-p\mathbf{t}]}{d\sigma} = \left[-\frac{pa}{\sqrt{\Delta a}} - \frac{(q\varphi)(\varphi a)}{\sqrt{\Delta a}} + q\Gamma_1 + p\Gamma_2\right](p\mathbf{t}+q\mathbf{t})$$
$$+ q^2\mathfrak{a} - 2pq\mathfrak{m} + p^2\mathfrak{b},$$

$$(19) \quad \frac{d[q\mathbf{t}-p\mathbf{t}]}{ds} = \left[\frac{(qa)}{\sqrt{\Delta a}} - \frac{(p\varphi)(\varphi a)}{\sqrt{\Delta a}} + p\Gamma_1 - q\Gamma_2\right](p\mathbf{t}+q\mathbf{t})$$
$$+ pq\mathfrak{a} + (q^2-p^2)\mathfrak{m} - pq\mathfrak{b},$$

$$\frac{d[p\mathbf{t}+q\mathbf{t}]}{d\sigma} = \left[\frac{(pa)}{\sqrt{\Delta a}} + \frac{(q\varphi)(\varphi a)}{\sqrt{\Delta a}} - q\Gamma_1 - p\Gamma_2\right](q\mathbf{t}-p\mathbf{t})$$
$$+ pq\mathfrak{a} + (q^2-p^2)\mathfrak{m} - pq\mathfrak{b}.$$

From these equations we have

$$(20) \quad \begin{aligned} \bar{\mathfrak{a}} &= \mathfrak{a}\cos^2\theta + 2\mathfrak{m}\sin\theta\cos\theta + \mathfrak{b}\sin^2\theta, \\ \bar{\mathfrak{b}} &= \mathfrak{a}\sin^2\theta - 2\mathfrak{m}\sin\theta\cos\theta + \mathfrak{b}\cos^2\theta, \\ \bar{\mathfrak{m}} &= (\mathfrak{a}-\mathfrak{b})\sin\theta\cos\theta + \mathfrak{m}(\sin^2\theta - \cos^2\theta), \end{aligned}$$

where $\bar{\mathfrak{a}}$, $\bar{\mathfrak{b}}$, $\bar{\mathfrak{m}}$ for the new system of curves correspond to $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{m}$ of the old.

Again, if we denote by G_1 the geodesic curvature of the curves having the tangent vectors $p\mathbf{t}+q\mathbf{t}$, and by G_2 the geodesic curvature of the curves having the tangent vectors $q\mathbf{t}-p\mathbf{t}$ we have the formulas

$$(21) \quad \begin{aligned} G_1 &= -\frac{(qa)}{\sqrt{\Delta a}} + \frac{(p\varphi)(\varphi a)}{\sqrt{\Delta a}} - p\Gamma_1 + q\Gamma_2, \\ G_2 &= -\frac{(pa)}{\sqrt{\Delta a}} - \frac{(q\varphi)(\varphi a)}{\sqrt{\Delta a}} + q\Gamma_1 + p\Gamma_2. \end{aligned}$$

13. **Planar curvature.** The derivative of the unit two-dimensional vector, tangent to the surface at a point of the curve, taken with respect to arc length along this curve, is connected with the curvature of the curve itself and also with the well known Gaussian curvature of the surface. We shall call this quantity the planar curvature of the curve with respect to the surface f.

We have seen that the expression $(f\varphi)$ represents a two-dimensional vector tangent to the surface, and that $(f\varphi)^2 = 2!$; hence, $\sqrt{\tfrac{1}{2}}(f\varphi)$ is the unit tangent two-dimensional vector.

Denoting the planar curvature vector by k_p we have

$$k_p = \frac{((f\varphi)a)}{\sqrt{2}\sqrt{\Delta_1 a}}.$$

The magnitude of this vector is the planar curvaturve itself. Denoting it by k we have,

$$k^2 = \frac{1}{2\Delta_1 a}((f\varphi)a)((f\varphi)a).$$

The connection of this with the curvature vectors previously studied, and the Gaussian curvature, is found by multiplying by $(\psi\theta)^2$ with the use of rules A and B:

$$\begin{aligned} 4k^2\Delta_1 a &= ((f\varphi)a)(\psi\theta)((f\varphi)a)(\psi\theta) \\ &= [((f\varphi)\psi)(a\theta) + ((f\varphi)\theta)(\psi a)]((f\varphi)a)(\psi\theta) && \text{(Rule A)} \\ &= 2((f\varphi)\psi)((f\varphi)a)(a\theta)(\psi\theta) \\ &= 2[((f\psi)\varphi) + ((\psi\varphi)f)]((f\varphi)a)(\psi\theta)(a\theta) && \text{(Rule B)} \\ &= 4((f\psi)\varphi)((f\varphi)a)(\psi\theta)(a\theta) \\ &= 4((f\psi)\psi)(\varphi\theta)(a\theta)((f\varphi)a) + 4((f\psi)\theta)(\psi\varphi)((f\varphi)a)(a\theta) && \text{(Rule A).} \end{aligned}$$

The first term of this expression is evidently equivalent to

$$4(\mathfrak{a}+\mathfrak{b})\mathfrak{a}(\theta a)^2.$$

The second term can be reduced as follows:

$$((f\psi)\theta)(\psi\varphi)((f\varphi)a)(a\theta) = ((f\psi)a)((f\varphi)\theta)(\psi\varphi)(a\theta) - ((f\psi)(f\varphi))(\psi\varphi)(a\theta)^2.$$

The first term on the right is the negative of the term on the left and the second term on the right contains the factor $((f\psi)(f\varphi))(\psi\varphi)$, which Maschke has shown to be twice the Gaussian curvature.

Substituting these results into the expression for $4\Delta_1 ak^2$ and dividing through by $\Delta_1 a = (\theta a)^2$ we have the result

$$k^2 = \mathfrak{a}(\mathfrak{a}+\mathfrak{b}) - K.$$

The square of the planar curvature in the orthogonal direction is clearly $\mathfrak{b}(\mathfrak{a}+\mathfrak{b}) - K$ and the sum of the squares of the curvatures in the two orthogonal directions is therefore equal to $4h^2 - 2K$ which is independent of the curve $a = \text{const.}$; later it will be seen that this invariant may be written $\frac{1}{2}((f\varphi)\psi)((f\varphi)\psi)$.

14. Principal directions for planar curvature. The expression for

$$k^2 = \frac{1}{2}\frac{\overline{d(f\varphi)}^2}{ds^2}$$

can be put in another useful form; $d(f\varphi)$ can be written $d(f\varphi) = (f\varphi)_1\, du + (f\varphi)_2\, dv$ and from this we obtain

$$k^2 = \frac{\mathfrak{E}\, du^2 + 2\mathfrak{F}\, du\, dv + \mathfrak{G}\, dv^2}{E\, du^2 + 2F\, du\, dv + G\, dv^2},$$

where

$$\mathfrak{E} = \frac{1}{2}(f\varphi)_1^2, \qquad \mathfrak{F} = \frac{1}{2}(f\varphi)_1\,(f\varphi)_2, \quad \text{and} \quad \mathfrak{G} = \frac{1}{2}(f\varphi)_2^2.$$

These quantities can evidently be expressed in terms of any of the sets which have been referred to as second fundamental quantities.

The values of $\lambda = dv/du$ which make k^2 a maximum or minimum are found, in the usual manner, to satisfy the equation

$$(E\mathfrak{F} - \mathfrak{E}F) + (E\mathfrak{G} - \mathfrak{E}G)\lambda + (F\mathfrak{G} - \mathfrak{F}G)\lambda^2 = 0.$$

The corresponding values of $R = 1/k^2$, also found in the usual way, are given by the equation

$$(\mathfrak{E}\mathfrak{G} - \mathfrak{F}^2)R^2 - (G\mathfrak{E} - 2F\mathfrak{F} + E\mathfrak{G})R + EG - F^2 = 0.$$

If $a =$ const. is the equation of a curve whose direction at each point coincides with the direction determined by one of the values of λ obtained from the above equation, then a satisfies the differential equation

$$(f(\varphi\,\psi))\,(fa)\,((\varphi\,\psi)\,a) = 0.$$

This equation may be reduced as follows:

$$\begin{aligned}(f(\varphi\,\psi))\,(fa)\,((\varphi\,\psi)\,a) &= (\varphi\,(f\psi))\,(fa)\,((\varphi\,\psi)\,a) + (\psi(\varphi f))\,(fa)\,((\varphi\,\psi)\,a)\\ &= 2\,(\varphi\,(f\psi))\,(fa)\,((\varphi\,\psi)\,a)\\ &= 2\,(f(f\psi))\,(\varphi\,a)\,((\varphi\,\psi)\,a) + 2\,(a(f\psi))\,(f\varphi)\,((\varphi\,\psi)a).\end{aligned}$$

The second term of this last expression vanishes so that the equation reduces to

$$(f(f\psi))\,(\varphi\,a)\,((\varphi\,\psi)\,a) = 0,$$

or

$$\boldsymbol{h}\cdot\mathbf{m} = 0.$$

The two directions defined by the equation in λ are orthogonal as may be seen by applying the test usually given in ordinary differential geometry, which is equally valid here.

In the equation for R put f_1^2, f_2^2, and $f_1 f_2$ for E, F, and G, respectively, and $\frac{1}{2}(f\varphi)_1^2$, $\frac{1}{2}(f\varphi)_1(f\varphi)_2$, and $\frac{1}{2}(f\varphi)_2^2$ for $\mathfrak{E}$, $\mathfrak{F}$, $\mathfrak{G}$. The equation then becomes

$$\frac{1}{2}\beta(\theta\psi)_1(f\varphi)_2((\theta\psi)(f\varphi))R^2 - \frac{1}{2}(f(\varphi\psi))^2 R + 1 = 0.$$

The coefficient of R^2 may also be written

$$-\beta(f\varphi)_1(\theta\psi)_2((\theta\psi)(f\varphi)),$$

as is seen by interchanging f with θ and ψ with φ. Thus the coefficient of R^2 may finally be written

$$\frac{1}{2}((\theta\psi)(f\varphi))^2,$$

and the equation for R becomes

$$\frac{1}{4}((\theta\psi)(f\varphi))^2 R^2 - \frac{1}{2}(f(\varphi\psi))^2 R + 1 = 0.$$

For surfaces in ordinary space the directions for the maximum and minimum of $[d(f\varphi)/ds]^2$ coincide with the principal directions and

$$\frac{\mathfrak{E}\mathfrak{G} - \mathfrak{F}^2}{EG - F^2} = K.$$

15. Various differential parameters and invariants. It is possible to write down immediately, in the symbolic notations, a number of invariants. Some of these will involve only the first fundamental quantities and their derivatives. Others will involve the second fundamental quantities as well. In two dimensions the only invariant of the first of these types is the Gaussian curvature, K. Maschke has given the formula

$$K = \frac{1}{2}((\varphi f)(\psi f))(\varphi\psi).$$

The invariants

$$((f\varphi)\varphi)((f\psi)\psi) = 4h^2, \quad \text{and} \quad ((f\varphi)\psi)((f\varphi)\psi)$$

are examples of invariants which depend upon the second as well as the first fundamental quantities.

Various scalar products formed from the vectors $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{m}$ furnish examples of differential parameters of the second order depending on one arbitrary function a of the coördinates.

There exist a number of interesting relations connecting these invariants and differential parameters and some of these will now be derived.

(a) The invariant

$$((f\varphi)\psi)((f\varphi)\psi)$$

may be written

$$\begin{aligned}[((f\psi)\varphi)+((\psi\varphi)f)]((f\varphi)\psi) &= 2((f\psi)\varphi)((f\varphi)\psi)\\ &= 2((f\psi)(f\varphi))(\varphi\psi)+2((f\psi)\psi)((f\varphi)\varphi)\\ &= -4K+8\mathbf{h}^2.\end{aligned}$$

(b) From formulas (15) we have

$$\begin{aligned}\frac{1}{\Delta_1 a^2}\,\mathfrak{a}\cdot\mathfrak{b} &= (\psi\varphi)(\psi a)((f\varphi)a)(\varrho a)(pa)((f\varrho)\theta)\\ &= (\psi\varphi)(\theta a)((f\varphi)a)(\varrho a)((f\varrho)\psi)(\theta a)+(\psi\varphi)(\theta a)((f\varphi)a)(\varrho a)((f\varrho)a)(\psi\theta).\end{aligned}$$

The last term reduces to $(\Delta_1 a)^2\mathfrak{m}^2$, since $(\psi\varphi)(\psi\theta)(\theta a) = (\varphi a)$. The first term can be reduced by similar methods to $(\Delta_1 a)^2 K$. Thus we obtain the formula

$$(\Delta_1 a)^2\mathfrak{a}\cdot\mathfrak{b} = (\Delta_1 a)^2 K+(\Delta_1 a)^2\mathfrak{m}^2,$$

or

$$K = \mathfrak{a}\cdot\mathfrak{b}-\mathfrak{m}^2.$$

III. Geometry of n-dimensional manifolds

16. **Curvatures of an orthogonal system of curves.** We consider now an arbitrary n-dimensional space defined by the vector $f(u_1, u_2, u_3, \cdots, u_n)$.

Let n subspaces each of $n-1$ dimensions mutually orthogonal at a point P be defined by the system of equations

$$a^1 = \text{const.},\qquad a^2 = \text{const.},\ \cdots,\ a^n = \text{const.} \tag{22}$$

Any system of $n-1$ of the equations of the system (22) defines a curve in the space f. The tangent vectors to these curves are

$$(fa^1a^2\cdots a^{i-1}a^{i+1}\cdots a^n),$$

and these vectors are mutually orthogonal.* We write

$$\mathfrak{t}^i = \frac{(fa)}{\sqrt{(fa)^2}} \tag{23}$$

where (fa) is an abbreviation for $(fa^1a^2\cdots a^{i-1}a^{i+1}\cdots a^n)$.

* It will be understood that the discussion relates to the point P mentioned above.

The unit tangent vector t^i may also be written in the form

$$t^i = \frac{(f\varphi)(\psi\varphi)(\psi a)}{(n-1)!\sqrt{(fa)^2}}. \tag{24}$$

It will be convenient to denote $(fa)^2$ by Δa. Using this abbreviation we write down at once, by formula (11), the expression for dt^i/ds_i:

$$\begin{aligned}\frac{dt^i}{ds_i} = \frac{(f\varphi)(\psi\varphi)((\psi a)a)}{(n-1)!\,\Delta a} + \frac{(\psi\varphi)(\psi a)((f\varphi)a)}{(n-1)!\,\Delta a}\\ - \frac{(\psi a)((\psi a)a)}{(n-1)!\,(\Delta a)}(f\varphi)(\theta\varphi)(\theta a).^*\end{aligned} \tag{25}$$

The first term of this expression is tangent to the space f and may be expressed linearly in terms of the vectors (fc) where c represents the various combinations of $n-1$ of the functions a. One of the terms of this expansion (the one for which the corresponding combination c has a^i missing) will just cancel the negative term at the end and we have the result

$$\frac{dt^i}{ds_i} = \sum_c^{\prime} \frac{(\psi c)((\psi a)a)}{\Delta c\,\Delta a}(fc) + \frac{(\psi\varphi)(\psi a)((f\varphi)a)}{(n-1)!\,\Delta a}, \tag{26}$$

where the mark $(\prime)$ above the summation sign indicates that the term in which the combination c is identical with the combination a is to be omitted.

The fact that the term (fa) is missing shows that the tangential component of the curvature vector dt^i/ds_i is orthogonal to t^i. This is, of course, to be expected.

17. Relations among the normal components. The normal components

$$\mathfrak{a}^i = \frac{(\psi\varphi)(\psi a)((f\varphi)a)}{(n-1)!\,\Delta a}$$

of the various curvature vectors dt^i/ds_i can be modified in the following manner.

Omit the denominator temporarily and suppose that the combination b is the one in which the function a^k is missing, and c is the combination in which a^j is missing. We may write

$$(\psi\varphi)(\psi b)((f\varphi)b) = \frac{(\psi\varphi)(\psi b)((f\varphi)b)(a)}{(a)},$$

---

* The term containing the product $(f\varphi)(\psi a)((\psi\varphi)a)$ vanishes. See Maschke, *Invariants*, loc. cit., formula (33).

where in the factor (a), of course, all the a's are present. Now

$$(\psi\varphi)(a) = \sum_c (a^j\varphi)(\psi c) = \sum_j (a^j\varphi)(a\,\psi\,a),$$

where ψ in the last factor occupies the jth position.

If the terms of the sum on the right are multiplied by (ψb) they will all vanish except the one for which $j = k$ since the vectors (ψb) and (ψc) are orthogonal; thus,

$$(27)\qquad \begin{aligned} \frac{(\psi\varphi)(\psi b)((f\varphi)b)}{(n-1)!\,\Delta b} &= (-1)^{k-1}\frac{(a^k\varphi)(\psi b)^2((f\varphi)b)}{(n-1)!\,\Delta b(a)} \\ &= (-1)^{k-1}\frac{(a^k\varphi)((f\varphi)b)}{(n-1)!\,(a)} = \mathfrak{a}^k. \end{aligned}$$

This gives, then, a new expression for the normal components of the curvature vector $d\mathbf{t}^k/ds_k$.

We now run the a^k of the first factor through the last factor and obtain

$$\mathfrak{a}^k = \frac{((f\varphi)\varphi)}{(n-1)!} + \sum_j (-1)^{k-1}\frac{(a^j\varphi)((f\varphi)\,a\,a^k\,a)}{(n-1)!\,(a)},$$

where in the last factor of the numerator the a^k occupies the jth position. To bring a^k to its proper position we must pass it over $k-j-1$ letters if $k>j$ or over $j-k-1$ letters if $k<j$. The resulting effective power of (-1) is in either case equal to j so that the jth term is the negative of $\mathfrak{a}^j$; hence we have

$$(28)\qquad \sum_i \mathfrak{a}^i = \frac{((f\varphi)\varphi)}{(n-1)!}.$$

18. **The cross curvatures.** By a procedure similar to that of § 11 we obtain the formulas for $d\mathbf{t}^i/ds_k$.

Let a denote the combination corresponding to $\mathbf{t}^i$ and b the combination corresponding to $\mathbf{t}^k$; also let c denote any combination. We have then

$$(29)\qquad \frac{d\mathbf{t}^i}{ds_k} = \sum_c^{\prime} \frac{(\psi c)((\psi a)b)}{\Delta c\sqrt{\Delta a}\sqrt{\Delta b}}(fc) + \frac{(\psi\varphi)(\psi a)((f\varphi)b)}{(n-1)!\sqrt{\Delta a}\sqrt{\Delta b}}.$$

This formula exhibits the curvature vector $d\mathbf{t}^i/ds_k$ as the sum of two components, one normal to the space f and the other tangent to f. If a and b are interchanged we have the formula for $d\mathbf{t}^k/ds_i$. We shall now show that the normal components of these two vectors are the same.

Consider the special case in which a is the combination $a^2 \cdots a^n$ and b is the combination $a^1\, a^3 \cdots a^n$. In the product $(\psi a)\,((f\varphi)\,b)$ run the a^2 of the first factor through the second. The first factor of each term of the result, except the first two, will contain a repeated a, and we may write

$$(\psi\varphi)\,(\psi a)\,((f\varphi)\,b) = (\psi\varphi)\,(\psi(f\varphi)\,a^3 \cdots a^n)\,(a^2\,a^1\,a^3 \cdots a^n) + (\psi\varphi)\,(\psi b)\,((f\varphi)\,a).$$

The first expression on the right vanishes* and the second is the result of interchanging a and b in the expression on the left. The two normals corresponding to these choices of the combinations a and b are thus seen to be the same. The same proof can be used for every choice of a and b.

The expressions $(\psi c)\,((\psi a)\,b)$ which occur in the coefficients of (fc) in the expression above for dt^i/ds_k are important differential parameters. From the formula $(\psi c)\,(\psi a) = 0$ we derive the equation

$$(\psi c)\,((\psi a)\,b) + (\psi a)\,((\psi c)\,b) = 0,$$

which is an important relation connecting these quantities. These quantities are closely related to Ricci's coefficients of rotation between which there exist relations of precisely the same nature as the one just obtained.

19. **Generalization of planar curvature.** The generalized planar curvature in n dimensions is defined in a manner altogether analogous to that used for the corresponding idea in two dimensions.

The tangent n-dimensional space may be written (f) as in the two-dimensional case, and its magnitude is determined from the relation

$$(f)^2 = n!.$$

The unit n-dimensional tangent vector is therefore

$$T = \frac{1}{\sqrt{(n!)}}(f).$$

The derivative of T with respect to arc length in any given direction may be taken as the (vector) planar curvature in that direction. The numerical value may, as usual, be computed from the scalar square of the vector curvature.

* See Maschke, loc. cit., formula 52, p. 455.

If we write $\mathfrak{E}_{ij} = (1/n!)(f)_i(f)_j$, we have at once

$$\left[\frac{1}{\sqrt{n!}}\frac{d(f)}{ds}\right]^2 = \frac{\sum_j\sum_i \mathfrak{E}_{ij}\,du_i\,du_j}{\sum_j\sum_i E_{ij}\,du_i\,du_j} = \frac{1}{R}. \tag{30}$$

If $du_1\cdots du_n$ are proportional to the variables $\lambda_1\cdots\lambda_n$ the necessary conditions for maximum or minimum values of $1/R$ may be written

$$\left(\sum_j\sum_i E_{ij}\lambda_i\lambda_j\right)\left(\sum_i \mathfrak{E}_{ik}\lambda_i\right) - \left(\sum_j\sum_i \mathfrak{E}_{ij}\lambda_i\lambda_j\right)\left(\sum_i E_{ik}\lambda_i\right) = 0,$$

and hence for these values of R

$$\frac{1}{R} = \frac{\sum_i \mathfrak{E}_{ik}\lambda_i}{\sum_i E_{ik}\lambda_i},$$

and hence also

$$\sum_i (E_{ik} - R\mathfrak{E}_{ik})\lambda_i = 0. \tag{31}$$

The elimination of the variables λ_i from these equations leads to an equation of the nth degree in R which we write in the form

$$1 + J_1 R + J_2 R^2 + \cdots + J_n R^n = 0.$$

The coefficients J can be expressed symbolically. Replace E_{ik} and $\mathfrak{E}_{ik}$ by $f_i f_k$ and $(1/n!)(f)_i(f)_k$ respectively, and reduce by the method of Bates (loc. cit., p. 27):

$$J_m = \frac{1}{n!}\,\frac{1}{n!\,m!\,(n-m)!}\left((f)^1(f)^2\cdots(f)^m f\right)^2, \tag{32}$$

where $(f)^1\cdots(f)^m$ each contain n equivalent symbols distinct from one another and also distinct from the remaining f's which fill out the invariantive constituent.

As in Bates, formula 29, p. 24, we may write equation (31) in the form

$$f_k\sum_i f_i\,du_i = R(f)_k\sum_i (f)_i\,du_i$$

and also in the form

$$f_k(fa) = R(f)_k((f)a). \tag{33}$$

20. **Principal directions. Lines of curvature.** The directions which correspond to the stationary values of R, when they are all distinct, are the principal directions. A line of curvature is a curve whose direction at each point coincides with a principal direction.

For two different lines of curvature

$$(34)\qquad \begin{aligned} (f)_k((f)a) &= \frac{1}{R_1} f_k(fa), \\ (f)_k((f)b) &= \frac{1}{R_2} f_k(fb). \end{aligned}$$

Now the cofactor of $(f)_i$ in the determinant $((f)a)$ is the same as the cofactor of f_i in (fa) since they both depend in the same way on the functions a; hence if we multiply the first of the above equations by the cofactors of f_k in the determinant (fb) and add the results for $k = 1, 2, \cdots, n$, we have

$$(35)\qquad ((f)a)((f)b) = \frac{1}{R_1}(fa)(fb);$$

and similarly from the second of the above

$$(36)\qquad ((f)a)((f)b) = \frac{1}{R_2}(fa)(fb);$$

hence if R_1 and R_2 are different we must have

$$(37)\qquad (fa)(fb) = 0,$$

and the corresponding lines of curvature are orthogonal.

University of Missouri,
Columbia, Mo.

ERRATA, VOLUME 26

B. A. BERNSTEIN, *Operations with respect to which the elements of a boolean algebra form a group.*

On page 171, after postulate P_2 insert: "P_2'. For any two elements a, b in K there exists an element y such that $y \circ a = b$", and in line 20 for "P_1, P_2" read "P_1, P_2, P_2'." Make changes in later paragraphs accordingly, omitting Theorem 1, in Theorem 2 substituting "a" for "an abelian," and adding: "All groups are abelian."

R. G. D. RICHARDSON, *Relative extrema of pairs of quadratic and hermitian forms.*

Page 479, formula (2), insert "$= 0$" after "$\sum_j (a_{ij} - \lambda b_{ij}) x_j$."